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# Heisenberg Uncertainty Principle for Momentum and Position Coordinates Referred to the Joule-Lenz Relation for Energy and Time 

Stanislaw Olszewski

## ABSTRACT

The quantum aspects of the Joule-Lenz law for the dissipation of energy imply for small transition energies the validity of the formula $\Delta \mathrm{E} \Delta \mathrm{t}=\mathrm{h}$ where $\Delta \mathrm{t}$ is the transition time of an electron particle. The present paper points out that a similar relation $\Delta p_{x} \Delta \mathrm{x}=h$ is valid for a particle enclosed in a one-dimensional potential box on condition the momentum change $\Delta p_{x}$ concerns two neighbouring quantum states. A corresponding change of the particle position coordinate $\Delta \mathrm{x}$ can be not so small.

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# Heisenberg Uncertainty Principle for Momentum and Position Coordinates Referred to the JouleLenz Relation for Energy and Time 

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## I. ABSTRACT

The quantum aspects of the Joule-Lenz law for the dissipation of energy imply for small transition energies the validity of the formula $\Delta \mathrm{E} \Delta \mathrm{t}=h$ where $\Delta \mathrm{t}$ is the transition time of an electron particle. The present paper points out that a similar relation $\Delta p_{x} \Delta \mathrm{x}=h$ is valid for a particle enclosed in a one-dimensional potential box on condition the momentum change $\Delta p_{x}$ concerns two neighbouring quantum states. A corresponding change of the particle position coordinate $\Delta \mathrm{x}$ can be not so small.

The relation for $\Delta p_{x}$ and $\Delta \mathrm{x}$ obtained for the onedimensional case can be found to exist also in a three-dimensional system of the hydrogen atom taken as an example.

Keywords: dissipation time and energy of electron transitions, reference of the momentum intervals to position intervals in transitions.
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## II. INTRODUCTION

The time intervals $\Delta t$ seem to appear very seldom in a quantum-mechanical practice; see e.g. [1-3]. The main source of application of $\Delta t$ is the Heisenberg uncertainty principle for energy and time expressed by the formula $[1,4]$

$$
\begin{equation*}
\Delta E \Delta t>\hbar \tag{1}
\end{equation*}
$$

However this formula and its physical background have been objected on many
occasions [5-7]. In fact a modification of (1) has been proposed by the expression [8-10]

$$
\begin{equation*}
2 m c^{2} \Delta E(\Delta t)^{2}>\hbar^{2} \tag{2}
\end{equation*}
$$

The particle (electron) mass $m$ entering (2) has been assumed to be a constant, so (2) is really a non-relativistic formula, nevertheless in its derivation $[8,10]$ the condition

$$
\begin{equation*}
v<c \tag{3}
\end{equation*}
$$

coupling the particle velocity $v$ and the speed of light $c$ has been taken into account.

More recently - in effect of a quantum analysis of the Joule-Lenz law for the dissipation of energy a proposal of the kind of (1) has been done in the form [11-15]

$$
\begin{equation*}
\Delta E \Delta t=h \tag{4}
\end{equation*}
$$

The validity of (4) was checked for simple quantum systems in which the energy interval $\Delta \mathrm{E}$ between two neighbouring quantum levels $n+1$ and $n$ has been examined together with $\Delta t$ which is the emission time of energy from $n+1$ to $n$.

Simultaneously - according to the Heisenberg uncertainty relations [1, 4] - the momentum changes

$$
\begin{equation*}
\Delta p_{x}, \Delta p_{y}, \Delta p_{z} \tag{5}
\end{equation*}
$$

and changes of the Cartesian coordinates

$$
\begin{equation*}
\Delta x, \Delta y, \Delta z \tag{6}
\end{equation*}
$$

are coupled together into expressions

$$
\begin{align*}
& \Delta p_{x} \Delta x>\hbar  \tag{7}\\
& \Delta p_{y} \Delta y>\hbar  \tag{7a}\\
& \Delta p_{z} \Delta z>\hbar \tag{7b}
\end{align*}
$$

The (7), (7a) and (7b) are considered to be actually valid formulae.

The aim of the present paper is to check the validity of (7), (7a) and (7b) in consequence of validity of the formula (4).

## III. ANALYSIS OF THE QUANTUM-

MECHANICAL MOTION IN ONE DIMENSION

Since

$$
\begin{equation*}
\hbar=\frac{h}{2 \pi}, \tag{8}
\end{equation*}
$$

so

$$
\begin{equation*}
\hbar<h, \tag{9}
\end{equation*}
$$

there does exist no contradiction between (1) and (4). Nevertheless (4) provides us with a much more accurate information than (1).

Classically the energy of a free electron in one dimension having the momentum $p_{x}$ is

$$
\begin{equation*}
E=\frac{p_{x}^{2}}{2 m} \tag{10}
\end{equation*}
$$

and the energy change is

$$
\begin{equation*}
\Delta E=\frac{2 p_{x} \Delta p_{x}}{2 m}=\frac{p_{x} \Delta p_{x}}{m} \tag{11}
\end{equation*}
$$

The quantum-mechanical energy is [16]

$$
\begin{equation*}
E=E_{n}=\frac{n^{2} h^{2}}{8 m L^{2}} \tag{12}
\end{equation*}
$$

Since

$$
\begin{equation*}
\Delta E \Delta t=\Delta x \Delta p_{x} \tag{15}
\end{equation*}
$$

Therefore the formulae (4) and (15) imply

$$
\begin{equation*}
\Delta p_{x} \Delta x=h \tag{16}
\end{equation*}
$$

Consequently to (15)

$$
\begin{equation*}
p_{x}=m \frac{\Delta x}{\Delta t}, \tag{13}
\end{equation*}
$$

we obtain for (11):

$$
\begin{equation*}
\Delta E=\frac{m \Delta x \Delta p_{x}}{\Delta t m}=\frac{\Delta x \Delta p_{x}}{\Delta t} \tag{14}
\end{equation*}
$$

or
$\square$

$$
\begin{equation*}
\frac{\Delta x}{\Delta t}=\frac{\Delta E}{\Delta p_{x}} \tag{17}
\end{equation*}
$$

which in fact is the Hamilton equation for the particle velocity.
The quantum-mechanical velocity $v_{x n}$ satisfies

$$
\begin{equation*}
E_{n}=\frac{m}{2} v_{x n}^{2} \tag{18}
\end{equation*}
$$

so from (12)

$$
\begin{equation*}
v_{x n}=\left(\frac{2 E_{n}}{m}\right)^{1 / 2}=\frac{n h}{2 m L} \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
p_{x n}=m v_{x n}=\frac{n h}{2 L} \tag{20}
\end{equation*}
$$

This gives

$$
\begin{equation*}
\Delta p_{x}=\Delta p_{x n}=[(n+1)-n] \frac{h}{2 L}=\frac{h}{2 L} \tag{21}
\end{equation*}
$$

In effect of (16) and (21)

$$
\begin{equation*}
\Delta x \Delta p_{x n}=\Delta x \frac{h}{2 L}=h \tag{22}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\Delta x=2 L \tag{23}
\end{equation*}
$$

On the other hand from (12)

$$
\Delta E=\Delta E_{n}=\frac{(n+1)^{2}-n^{2}}{8 m L^{2}} h^{2} \cong \frac{2 n}{8 m L^{2}} h^{2}=\frac{n h^{2}}{4 m L^{2}} .
$$

This expression divided by $\Delta p_{x}$ from (21) gives

$$
\begin{equation*}
\frac{\Delta E}{\Delta p_{x}}=\frac{n h^{2}}{4 m L^{2}} \cdot \frac{2 L}{h}=\frac{n h}{2 m L}=v_{x n}=\frac{\Delta x}{\Delta t} \tag{25}
\end{equation*}
$$

which is in a full agreement with the velocity expression (19).

Let us note that result in (25) could be obtained directly from the formula (15).

We conclude this section by noting that (7) replaced by (16) leads to a correct description of the motion parameters in a one-dimensional potential box. This holds on condition that solely a low-energy transition

$$
\begin{equation*}
n+1 \longrightarrow n \tag{26}
\end{equation*}
$$

is taken into account.

## IV. IMPLICATIONS CONCERNING A THREEDIMENSIONAL SYSTEM

Such system is given by the hydrogen atom. The electron is a particle bound to the atomic nucleus, so its energy is negative and equal to the electron kinetic energy

$$
\begin{equation*}
-E=\frac{p_{x}^{2}}{2 m}+\frac{p_{y}^{2}}{2 m}+\frac{p_{z}^{2}}{2 m} \tag{27}
\end{equation*}
$$

This second property is due to the virial theorem which states that the total energy of the $H$ atom is equal to the negative kinetic energy of that atom:

$$
\begin{equation*}
E=E_{\mathrm{tot}}=-E_{\mathrm{kin}} \tag{28}
\end{equation*}
$$

In the emission spectrum of energy a less negative E becomes a more negative number leading to an increase of the electron kinetic energy.

If we assume that in a transition process of energy only one momentum component of the electron is involved, we obtain

$$
\begin{equation*}
-\Delta E=\frac{p_{x} \Delta p_{x}}{m}=m \frac{\Delta x}{\Delta t} \frac{\Delta p_{x}}{m}=\frac{\Delta x \Delta p_{x}}{\Delta t} \tag{29}
\end{equation*}
$$

so

$$
\begin{equation*}
\Delta E \Delta t=-\Delta x \Delta p_{x} \tag{30}
\end{equation*}
$$

This gives

$$
\begin{equation*}
-\frac{\Delta E}{\Delta p_{x}}=\frac{\Delta x}{\Delta t} \tag{31}
\end{equation*}
$$

which is the electron velocity, and for the case

$$
\begin{equation*}
\Delta E \Delta t=h \tag{32}
\end{equation*}
$$

we obtain from (31)

$$
\begin{equation*}
\Delta p_{x} \Delta x=-h \tag{33}
\end{equation*}
$$

which differs from the free-electron case (16) solely in its sign.

All momentum components $p_{\mathrm{x}}, p_{y}$ and $p_{z}$ can be changed independently, therefore we obtain that

$$
\Delta E \Delta t=-\Delta p_{x} \Delta x=-\Delta p_{y} \Delta y=-\Delta p_{z} \Delta z=h
$$

Expression (34) implies a three-fold degeneracy of $\Delta \mathrm{E} \Delta \mathrm{t}$ with respect to the direction of momentum. In consequence (31) can be supplemented by the formulae

$$
\begin{align*}
& -\frac{\Delta E}{\Delta p_{y}}=\frac{\Delta y}{\Delta t}  \tag{35}\\
& -\frac{\Delta E}{\Delta p_{z}}=\frac{\Delta z}{\Delta t} \tag{36}
\end{align*}
$$

This is an expected result because the hydrogen atom is an isotropic system.

In fact (31), (35) and (36) play a role of the Hamilton equations for the components of the electron velocity.

## V. DISCUSSION

What will happen when (32) is not satisfied? This is a situation which concerns, for example, the intensity spectrum of the hydrogen atom in case when the electron transitions are performed between the non-neighbouring quantum levels. For the sake of definiteness let us assume

$$
\begin{equation*}
\Delta E=\Delta E_{1}+\Delta E_{2} \tag{37}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta E_{1}=E_{n+1}-E_{n}  \tag{38}\\
& \Delta E_{2}=E_{n+2}-E_{n+1} \tag{39}
\end{align*}
$$

The transition time corresponding to $\Delta \mathrm{E}$ is

$$
\begin{equation*}
\Delta t=\Delta t_{1}+\Delta t_{2} \tag{40}
\end{equation*}
$$

where $\Delta t_{1}$ is the transition time necessary for $\Delta \mathrm{E}_{1}$, so according to (4)

$$
\begin{equation*}
\Delta E_{1} \Delta t_{1}=h \tag{41}
\end{equation*}
$$

and $\Delta t_{2}$ is the transition time necessary for the emission of $E_{2}$, so

$$
\begin{equation*}
\Delta E_{2} \Delta t_{2}=h \tag{42}
\end{equation*}
$$

The last two formulae, (41) and (42), represent the quantum aspect of the J oule-Lenz law; they are satisfied especially well when the quantum number $n$ is large [8-11]. Because of (41) and (42) we obtain for (40):

$$
\begin{equation*}
\Delta t=\Delta t_{1}+\Delta t_{2}=\frac{h}{\Delta E_{1}}+\frac{h}{\Delta E_{2}} \tag{43}
\end{equation*}
$$

Our aim is to calculate the product
similar to (41) and (42). We obtain

$$
\Delta E \Delta t=\left(\Delta E_{1}+\Delta E_{2}\right) h\left(\frac{1}{\Delta E_{1}}+\frac{1}{\Delta E_{2}}\right)=\frac{(\Delta E)^{2} h}{\Delta E_{1} \Delta E_{2}} .(44 \mathrm{a})
$$

It is evident from (37)-(42) that the intervals $\Delta \mathrm{E}$ $>\Delta \mathrm{E}_{1}, \Delta \mathrm{E}_{2}$ as well as $\Delta \mathrm{t}>\Delta \mathrm{t}_{1}, \Delta \mathrm{t}_{2}$, so

$$
\Delta E \Delta t=\frac{(\Delta E)^{2}}{\Delta E} \Delta t>\frac{\Delta E_{1} \Delta E_{2}}{\Delta E} \Delta t=h
$$

obtained on the basis of (44a), or

$$
\begin{equation*}
-\Delta E \Delta t<-h \tag{45a}
\end{equation*}
$$

By assuming that also for the case of $\Delta \mathrm{E}$ and $\Delta \mathrm{t}$ entering (37) and (40) there is satified the Hamilton equation

$$
\begin{equation*}
-\frac{\Delta E}{\Delta p_{x}}=\frac{\Delta x}{\Delta t} \tag{46}
\end{equation*}
$$

we obtain - because of (45) and (45a) - the relation

$$
\begin{equation*}
-\Delta E \Delta t=\Delta p_{x} \Delta x<-h \tag{47}
\end{equation*}
$$

or

$$
\begin{equation*}
\Delta E \Delta t=-\Delta p_{x} \Delta x>h \tag{48}
\end{equation*}
$$

On the other hand, for the one-dimensional system we have the Hamilton equation

$$
\begin{equation*}
\frac{\Delta E}{\Delta p_{x}}=\frac{\Delta x}{\Delta t} \tag{49}
\end{equation*}
$$

but the relation (45) holds also in this case. Therefore on the basis of (45) and (49) we obtain for one dimension

$$
\begin{equation*}
\Delta E \Delta t=\Delta p_{x} \Delta x>h \tag{50}
\end{equation*}
$$

## VI. SUMMARY

The paper has been stimulated by an analysis of the Heisenberg uncertainty formulae coupling the changes of the Cartesian coordinates of the particle momentum and position. These formulae seem to be of a much more often use than their counterpart concerning the changes of energy and time.

In fact - which is demonstrated for simple quantum systems as well as in calculations of the intensity spectrum of the atomic hydrogen [17, 18] - small changes of energy and the time intervals connected with them satisfy an exact formula (4) which is not in opposition to the Heisenberg result but seems to have an advantage to provide us with more precise data than (1).

The present paper shows - by considering the motion of an electron particle in a onedimensional potential box and the electron of the hydrogen atom taken as examples - that the use of (4) can modify also the Heisenberg relations applied to coordinates of the momentum and position.

These relations - see (16) and (34) - could become of equal accuracy as (4) which is an accurate coupling of the intervals of energy and time for transition (26) and large n.

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