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M. A. Gopalan, K.Meena & S. Vidhyalakshmi

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In this communication, an attempt has been made to obtain pairs of non-zero distinct integers x, y such that, in each pair,

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A Study on two Curious Diophantine Problems

M. A .Gopalan [α], K.Meena [σ] & S. Vidhyalakshmi [ρ]

ABSTRACT

In this communication, an attempt has been made to obtain pairs of non-zero distinct integers x, y such that, in each pair,

i. *The square of the sum added with the cube of the sum is equal to two times the sum of the cubes of the corresponding integers.*
ii. *2k times the square of the sum added with the cube of the sum is equal to two times the sum of the cubes of the corresponding integers.*

Keywords: diophantine problems, binary cubic, integer solutions.

Author α: Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India.
σ: Former VC, Bharathidasan University, Trichy-620 024, Tamil Nadu, India.

I. INTRODUCTION

The subject of diophantine equations is one of the major areas in the history of Number Theory. It is obviously a broad topic and has a marvelous effect on credulous people and always occupy a remarkable position due to unquestioned historical importance because of the limitless supply of exciting non-routine and challenging problems [1-4]. All that needed is something to arouse interest and get started.

In this communication, an attempt has been made to obtain pairs of non-zero distinct integers x, y such that, in each pair,

(i) $(x+y)^2+(x+y)^3=2\left(x^3+y^3\right)$

(ii) $2k(x+y)^2+(x+y)^3=2\left(x^3+y^3\right)$

It is worth mentioning that in case (i), the process employs the integer solutions of the pellian equation $y^2=12x^2+1$ and in case (ii), the integer solutions of the positive pell equation $y^2=3x^2+k^2$ are employed.

II. METHOD OF ANALYSIS

Problem: 1

Let a, b be two distinct non-zero integers. The problem under consideration is mathematically equivalent to solving the binary cubic Diophantine equation represented by

$$(a+b)^2+(a+b)^3=2\left(a^3+b^3\right) \tag{1}$$

Introduction of linear transformations

$$a=u+v, b=u-v, (u \neq v \neq 0) \tag{2}$$

in (1) leads to the positive pellian equation

$$(2u+1)^2=12v^2+1$$

which is satisfied by

$$u_n=\frac{1}{4}(f_n-2), v_n=\frac{1}{4\sqrt{3}}g_n$$

where

$$f_n=\left(7+4\sqrt{3}\right)^{n+1}+\left(7-4\sqrt{3}\right)^{n+1}, \quad g_n=\left(7+4\sqrt{3}\right)^{n+1}-\left(7-4\sqrt{3}\right)^{n+1}$$

In view of (2), the values of a and b satisfying (1) are given by

$$a_n=\frac{1}{4}(f_n-2)+\frac{1}{4\sqrt{3}}g_n$$

$$b_n=\frac{1}{4}(f_n-2)-\frac{1}{4\sqrt{3}}g_n, \quad n=0,1,2,......$$

A few numerical examples are exhibited in Table: 1 below:

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Table 1: Numerical examples

n	a_n	b_n
0	5	1
1	76	20
2	1065	285
3	14840	3976

The recurrence relations satisfied by a_n , b_n are respectively presented below:

$$a_{n+2} - 14a_{n+1} + a_n = 6 \,,\; a_0 = 5 \,,\; a_1 = 76$$

$$b_{n+2} - 14b_{n+1} + b_n = 6 \,,\; b_0 = 1 \,,\; b_1 = 20$$

Observation 1:

From each of the values of a_n , $b_n(n \geq 2)$, one may generate second order Ramanujan Numbers.

Illustration:

Consider

$$a_2 = 1065 = 1065*1 = 5*213 = 15*71$$
$$= 533^2 - 532^2 = 109^2 - 104^2 = 43^2 - 28^2$$

Now,

$$533^2 - 532^2 = 109^2 - 104^2 \Rightarrow 533^2 + 104^2 = 532^2 + 109^2 = 294905$$
$$533^2 - 532^2 = 43^2 - 28^2 \Rightarrow 533^2 + 28^2 = 532^2 + 43^2 = 284873$$
$$109^2 - 104^2 = 43^2 - 28^2 \Rightarrow 109^2 + 28^2 = 104^2 + 43^2 = 12665$$

Thus, 294905, 284873, 12665 represent second order Ramanujan numbers

Observation: 2

It is seen that

$$a_n b_n + \frac{1}{48} g_n^2 = \frac{1}{16}(f_n - 2)^2 = r^2, \text{ say}$$

The pair (a_n, b_n) is a Diophantine 2-tuple with property $D\left(\frac{1}{48} g_n^2\right)$

Now,

$$a_n + b_n + 2r = f_n - 2$$

Note that

$$a_n(f_n - 2) + \frac{1}{48} g_n^2 = \left(\frac{1}{2}(f_n - 2) + \frac{1}{4\sqrt{3}} g_n\right)^2$$

$$b_n(f_n - 2) + \frac{1}{48} g_n^2 = \left(\frac{1}{2}(f_n - 2) - \frac{1}{4\sqrt{3}} g_n\right)^2$$

The above relations lead to the result that the triple $(a_n, b_n, f_n - 2)$ is a Diophantine 3-tuple with property $D\left(\frac{1}{48} g_n^2\right)$

Remark:

It is to be noted that, by considering either the pair $(a_n, f_n - 2)$ or $(b_n, f_n - 2)$, one generates two more Diophantine 3-tuples with property $D\left(\frac{1}{48} g_n^2\right)$. Thus, the repeated applications of the above process leads to the generation of sequence of diophantine 3-tuples with property $D\left(\frac{1}{48} g_n^2\right)$.

Problem: 2

We search for two distinct non-zero integers a, b such that

$$2k(a+b)^2 + (a+b)^3 = 2(a^3 + b^3) \,,\; k > 1 \qquad (3)$$

Introduction of linear transformations

$$a = u + v, b = u - v, (u \neq v \neq 0) \qquad (4)$$

in (3) leads to the positive pellian equation

$$y^2 = 3v^2 + k^2 \qquad (5)$$

where

$$y = u + k \qquad (6)$$

The fundamental solution of (5) is

$$v_0 = k \,,\; y_0 = 2k$$

To obtain the other solutions of (5), consider the pell equation

$$y^2 = 3v^2 + 1$$

whose solution is given by

$$\tilde{y}_n = \frac{1}{2} f_n \,,\; \tilde{v}_n = \frac{1}{2\sqrt{3}} g_n$$

where

$$f_n = \left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1} \,,\; g_n = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}$$

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Applying Brahmagupta lemma between (y_0, v_0) and $(\tilde{y}_n, \tilde{v}_n)$, the other integer solutions of (5) are given by

$$y_{n+1} = kf_n + \frac{\sqrt{3}}{2} k\, g_n$$
$$v_{n+1} = \frac{k}{2} f_n + \frac{k}{\sqrt{3}} g_n \quad , \quad n = -1, 0, 1, \ldots.$$

From (6), we get

$$u_{n+1} = k\, f_n + \frac{\sqrt{3}k}{2} g_n - k \quad , \quad n = -1, 0, 1, \ldots.$$

In view of (4), the values of a and b satisfying (3) are given by

$$a_{n+1} = \frac{3k}{2} f_n + \frac{5\sqrt{3}\,k}{6} g_n - k$$
$$b_{n+1} = \frac{k}{2} f_n + \frac{k\sqrt{3}}{6} g_n - k \quad , \quad n = 0, 1, 2, \ldots\ldots$$

Note: $a_{n+1} = b_{n+2}$

For properties, one has to go for particular values for k

Illustration:

Let $k = 1$. Then the corresponding values of a, b are given by

$$a_s = \frac{1}{2}(f_s - 2) + \frac{1}{2\sqrt{3}} g_s$$
$$b_s = \frac{1}{2}(f_s - 2) - \frac{1}{2\sqrt{3}} g_s \quad , \quad s = 0, 1, 2, \ldots\ldots$$

A few numerical examples are exhibited in Table: 2 below:

Table 2: Numerical examples

s	a_s	b_s
0	2	0
1	10	2
2	40	10
3	152	40
4	570	152

The recurrence relations satisfied by a_s , b_s are respectively presented below:

$$a_{s+2} - 4a_{s+1} + a_s = 2 \; , \; a_0 = 2 \, , \, a_1 = 10$$
$$b_{s+2} - 4b_{s+1} + b_s = 2 \; , \; b_0 = 0 \, , \, b_1 = 2$$

Observation: 3

From each of the values of $a_s (s \geq 2)$, $b_s (s \geq 3)$, one may generate second order Ramanujan Numbers.

Illustration:

Consider

$$a_2 = 40 = 2 * 20 = 4 * 10$$
$$= 11^2 - 9^2 = 7^2 - 3^2$$
$$\Rightarrow 11^2 + 3^2 = 9^2 + 7^2 = 130$$
$$a_3 = 152 = 2 * 76 = 4 * 38$$
$$= 39^2 - 37^2 = 21^2 - 17^2$$
$$\Rightarrow 39^2 + 17^2 = 37^2 + 21^2 = 1810$$

Thus, 130, 1810 represent second order Ramanujan numbers

Observation: 4

It is seen that

$$a_s b_s + \frac{1}{12} g_s^2 = \frac{1}{4}(f_s - 2)^2 = r^2, \quad \text{say}$$

The pair (a_s, b_s) is a Diophantine 2-tuple with property $D\left(\frac{1}{12} g_s^2\right)$

Now,

$$a_s + b_s + 2r = 2(f_s - 2)$$

Note that

$$2(f_s - 2)a_s + \frac{1}{12} g_s^2 = \left(f_s - 2 + \frac{1}{2\sqrt{3}} g_s\right)^2$$
$$2(f_s - 2)b_s + \frac{1}{12} g_s^2 = \left(f_s - 2 - \frac{1}{2\sqrt{3}} g_s\right)^2$$

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The above relations lead to the result that the triple $(a_s, b_s, 2(f_s - 2))$ is a Diophantine 3-tuple with property $D\left(\frac{1}{12}g_s^2\right)$

Following the remark presented in problem: 1, in this case also, a sequence of diophantine 3-tuples is generated with property $D\left(\frac{1}{12}g_s^2\right)$

In conclusion, the readers of this communication may search for other formulations of Diophantine problems involving different pellian equations.

REFERENCES

1. J.N. Kapur, Fascinating world of Mathematical Sciences, Vol.14, Mathematical Sciences Trust Society, New Delhi, 1994.
2. Shailesh Shirali, Mathematical Marvels, A primes on number sequences, Universities Press, India, 2001.
3. Titu Andreescu, Dorin Andrica and Zuming Feng, 104 Number Theory problems, Birkhauser Boston Inc., 2007.
4. Titu Andreescu and Dorin Andrica, Number Theory, Birkhauser Boston Inc., 2009.

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