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In this work, we consider the Minimum Maximal Network Flow Problem (MMNFP), i.e., minimizing the flow value, minimizing the total time among the maximal flow, and time windows, which is a combinatorial optimization and an NP-hard problem. We propose a new version of MMNFP; this version is a Minimum Maximal Network Fuzzy Flow Problem with Fuzzy Time-Windows (MMNFFPFTW). After a mathematical modeling problem, we introduce some basic definitions, basic formulations of the problem, and one of them is the minimization of the concave function over a fuzzy convex set. The problem can also study with the difference of the convex (nonconvex) functions programming. We propose a new algorithm of the MMNFFPFTW.

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I. INTRODUCTION

A Minimum Maximal Network Flow Problem (MMNFP) is a basic problem in the network flow theory with several applications, communication networks and logistic networks. In the last 20 years there are many active researchers in difference of a convex (nonconvex) functions programming, because most of real-life optimization problems are nonconvex see, ([8], [9] and [10]). The field of network optimization flows has a rich and long history, a difference of convex functions programming and the difference of the convex function algorithms introduced by Pham Donh Tao in 1985. The real-life early work established the foundation of the key ideas of the network optimization flow theory, see, ([1], [2], [3], [4] and [5]). The key task of this filed is to answer such questions as, which way to use the network of the most cost effective?

Iri [13] gave the definition of undirected flow (u-flow) and presented the fundamental problems related u-flow. Although the concept of u-flow is quite the different from maximal flow and their relationship is not known yet so much, the optimal value of the minimum maximal u-flow of the network flow G is equal to the best value of the minimum maximal flow under to sumption. In [13] profound essay, a several fundamental theorems and the maximal research topics are described, but no algorithms for the corresponding problems are proposed. To the author's knowledge, no algorithms for a minimum maximal flow were known until Shi-Yamamoto [20]. As pointed out in [24], Shi-Yamamoto's algorithm is not efficient enough. After that, some algorithms for solving the problem were proposed in such as Shigeno-Yamamoto [21] and others.

A maximum flow problem and a minimum cost flow problem are two typical problems of them, see, ([11], [12], [14] and [15]). However, from the point view of the practical cases, we have another kind of the problems which the different form of the typical ones is inherently. For instance, a Minimum Maximal Network Fuzzy Flow Problem with Fuzzy Time-Windows (MMNFFPFTW), by Figure 1 and 2 portrays a fuzzy network of the arc fuzzy flow capacity of one unit on all arcs, each arc has a transit fuzzy time $\tilde{t}_{v_i v_j}, \forall v_i, v_j \in V;\ (v_i, v_j) \in V, i \neq j; i, j = 1, \dots, n$, see, [6]. Each vertex $v_i \in V$ has a fuzzy time-windows $[\tilde{a}_{v_i}, \tilde{b}_{v_i}]$, within which the vertex may be served, i.e., $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}], \tilde{t}_{v_i v_j} \in \tilde{T}$, is a non-negative fuzzy service and leaving for that vertex. A source vertex s and a sink vertex τ with fuzzy time -windows $[\tilde{a}_s, \tilde{b}_s]$ and $[\tilde{a}_\tau, \tilde{b}_\tau]$ respectively, see, ([6], [7], [12] and [22]).

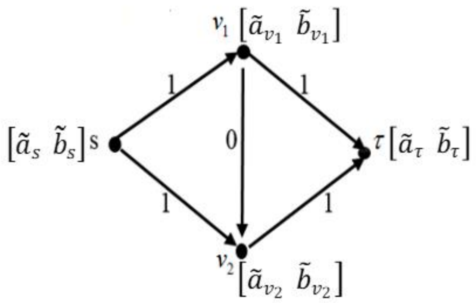

Figure 1

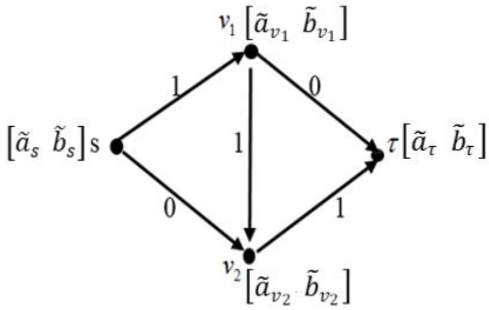

Figure 2

A Minimum Maximal Network Fuzzy Flow Problem with Fuzzy Time-Windows

The figure 1 illustrates the maximum fuzzy flow of the fuzzy network, that is, the fuzzy flow on all arcs is one except the arc $\tilde{x}_3$, whose fuzzy flow is zero. On the other hand, if the fuzzy flow on $\tilde{x}_3$ is fixed at one and we cannot reduce it by some reasons such as emergency, then the fuzzy network cannot be exploited at the most economical situation. In this case, we can send two unit of the fuzzy flow from a source vertex s to a sink vertex τ which satisfy the fuzzy time-windows constraint. In the figure 2, the fuzzy flow on $\tilde{x}_3$ is fixed at one, the possible fuzzy flow value, we can send between s and τ is one unit. The fuzzy flow value, we can send between s and τ reduces from two (in figure 1) to one (in figure 2) because the fuzzy flow value of $\tilde{x}_3$ is undirected. It means that the maximum fuzzy flow value is not attainable if the users on the fuzzy network are disobedient.

Form the point view of modeling, the above of two figures cases are essentially different though they bear some resemblance. If the fuzzy flow is directly, the figure 1 aims at an optimal value of the fuzzy flow. The figure 2 also, searches for an optimal value of the fuzzy flow, without a directly of the network fuzzy flow. The standard network fuzzy flow with directly has been well studied for several decades. Without a directly, many problems in network fuzzy flow, the maximum fuzzy flow problem, become more difficulty. Compared to the standard network fuzzy flow theory is a new filed, hence a still in its infancy, see, ([16], [17], [18], [19] and [23]).

The reminder of this work is organized as follows. In Section 2, we give the fuzzy concepts, mathematical models of the MMNFFPFTW and its equivalent formulations. In Section 3, we then outline the properties of the difference convex fuzzy programming and a difference convex fuzzy algorithm. We describe the framework of the difference convex fuzzy algorithm with the fuzzy time-windows. In Section 4, we give a new algorithm of the MMNFFPFTW. In the last section, the conclusion is given.

II. FUZZY CONCEPTS AND MATHEMATICAL MODELS

Consider a directed fuzzy network $\tilde{G} = \left(V, A, \tilde{t}_{v_i v_j}, [\tilde{a}_{v_i}, \tilde{b}_{v_i}]\right)$, where V is a set of n vertices, A is a set of n arcs with a non-negative transit fuzzy time $\tilde{t}_{v_i v_j}, \forall v_i, v_j \in V$, $(v_i, v_j) \in V, i \neq j; i, j = 1. \ldots . n$. For each vertex $v_i \in V$, has a fuzzy time-windows $[\tilde{a}_{v_i}, \tilde{b}_{v_i}]$ within which the vertex may be served with $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}]$ is a non-negative service and leaving fuzzy time of the vertex. A source vertex s, a sink vertex τ with a fuzzy time-windows $[\tilde{a}_s, \tilde{b}_s]$ and $[\tilde{a}_\tau, \tilde{b}_\tau]$ respectively, see, ([5], [6] and [22]) and $\tilde{c}$ is a vector of the arc fuzzy capacity. Let $\tilde{X}$ denote a fuzzy set of the feasible fuzzy flow,

$$\tilde{X} = \{\tilde{x} : \tilde{x} \in \mathcal{R}^n, \tilde{A}\tilde{x} = 0; 0 \leq \tilde{x} \leq \tilde{c}\} \quad (1)$$

where the matrix $\tilde{A}$ stands for a vertex arc incident the relationship in the fuzzy network. Obviously, $\tilde{X}$ is a compact convex fuzzy set.

Let $\tilde{f}$ be a fuzzy flow value function, $\tilde{f}$ is assumed to be a fuzzy linear on $\tilde{X}$. For instant, it usually fined by,

$$\tilde{f}(\tilde{x}) = \sum_{v_i \in \delta^+(s)} \tilde{x}_{v_i} - \sum_{v_i \in \delta^-(s)} \tilde{x}_{v_i} \quad (2)$$

where $\delta^+(s)$ and $\delta^-(s)$ are the sets of arcs which leaves and enters the source vertex s, respectively.

- A Mathematical Formulation Model
- We define an instant model with the fuzzy function $\tilde{r}$ as;

$$\tilde{r}(\tilde{x}) = \widetilde{max}\{\tilde{e}(\tilde{y} - \tilde{x}) : \tilde{y} \geq \tilde{x}, \tilde{y} \in \tilde{X}\} \quad (3)$$

where, $\tilde{r}(\tilde{x}) \geq 0, \forall \tilde{x} \in \tilde{X}$, $\tilde{e}$ denoted to both a row fuzzy vector and a column fuzzy vector of ones. Moreover, $\tilde{r}(\tilde{x})$ is the piecewise fuzzy linear on $\tilde{X}$. In fact, adding a slack fuzzy vector $\tilde{z}$ such that,

$$\begin{pmatrix} \tilde{A} & 0 & 0 \\ \tilde{I} & \tilde{I} & 0 \\ \tilde{I} & 0 & -\tilde{I} \end{pmatrix}\begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{c} \\ \tilde{x} \end{pmatrix}, \tilde{z} \in \mathcal{R}_*^{2n} \Leftrightarrow \tilde{A}\tilde{y} = 0, \tilde{x} \leq \tilde{y} \leq \tilde{c} \tag{4}$$

where, $\mathcal{R}_*^{2n}$ denotes the fuzzy set of $2n-$ dimensional real column fuzzy vectors and $\tilde{A}$ is a fuzzy matrix stands for the vertex arc incident a relationship in the fuzzy network. Then for a given $\tilde{x} \in \mathcal{R}_*^n, \tilde{r}(\tilde{x})$ is a solution of the following fuzzy linear programming:

$$\widetilde{max}(\tilde{e}\tilde{y} - \tilde{e}\tilde{x})$$

subject to $$\begin{pmatrix} \tilde{A} & 0 & 0 \\ \tilde{E} & \tilde{E} & 0 \\ \tilde{E} & 0 & -\tilde{E} \end{pmatrix}\begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{c} \\ \tilde{x} \end{pmatrix}, \tilde{y} \geq 0, \tilde{z} \geq 0 \tag{5}$$

where $\tilde{E}$ is an $n \times n$ fuzzy matrix. As $\tilde{r}(\tilde{x})$ is a solution of the linear fuzzy maximization, we assume that,

$$\tilde{r}(\tilde{x}) = \tilde{c}_{\tilde{B}}\tilde{B}^{-1}\begin{pmatrix} 0 \\ \tilde{c} \\ \tilde{x} \end{pmatrix}\text{-}\tilde{e}\tilde{x} \tag{6}$$

where $\tilde{c}_{\tilde{B}}$ corresponding to the coefficient fuzzy vector of the objective fuzzy function, and $\tilde{B}$ is a basic fuzzy matrix of the problem (5) which satisfy a fuzzy time-windows constraints

$$\tilde{t}_{v_i} + \tilde{t}_{v_{i,}v_j} \leq \tilde{t}_{v_j}, \tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}], \tilde{t}_{v_j} \in [\tilde{a}_{v_j}, \tilde{b}_{v_j}], \tilde{t}_{v_i}, \tilde{t}_{v_{i,}v_j} \in \mathcal{R}^+, \forall v_i, v_j \in V \tag{7}$$

- We define a dual formulation model

In Philip [11], it follows that there exists a simplex $\widetilde{\Lambda} \subseteq \mathcal{R}_n$ (the set of $n-$ dimensional real fuzzy row vectors) such that the vector $\tilde{x}$ is a maximal fuzzy flow if and only if there exists $\tilde{\lambda} \in \widetilde{\Lambda}$ such that

$$\tilde{\lambda}\tilde{x} \geq \tilde{\lambda}\tilde{y}, \forall \tilde{y} \in \tilde{X} \tag{8}$$

Thus, the MMNFFPFTW can be formulated as:

$$\widetilde{min}\tilde{f}(\tilde{x})$$

subject to $$-\tilde{\lambda}(\tilde{y} - \tilde{x}) \geq 0, \forall \tilde{y} \in \tilde{X}, \tilde{\lambda} \in \widetilde{\Lambda}, \tilde{x} \in \tilde{X} \tag{9}$$

$$\tilde{t}_{v_i} + \tilde{t}_{v_{i,}v_j} \leq \tilde{t}_{v_j}, \tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}], \tilde{t}_{v_j} \in [\tilde{a}_{v_j}, \tilde{b}_{v_j}], \tilde{t}_{v_i}, \tilde{t}_{v_{i,}v_j} \in \mathcal{R}^+, \forall v_i, v_j \in V \quad (10)$$

This is a special case of the mathematical fuzzy programming of a variation inequality with the fuzzy time-windows constraint. We denote that,

$$\tilde{g}(\tilde{x}, \tilde{\lambda}) = {}^1/_2\,\|\tilde{x}\|^2 + {}^1/_2\,\|\tilde{\lambda}\|^2 + \widetilde{max}_{\widetilde{w} \in \tilde{X}}\{\widetilde{w}\tilde{x} + \widetilde{w}\tilde{\lambda} - {}^1/_2\,\|\widetilde{w}\|^2\} \tag{10}$$

and $$\tilde{h}(\tilde{x}, \tilde{\lambda}) = {}^1/_2\,\|\tilde{x} + \tilde{\lambda}\|^2 + {}^1/_2\,\|\tilde{x}\|^2 \tag{11}$$

,

Then, we proof the following lemma

Lemma 2.1 The constraints (9) can be cast into the form:

$$\tilde{g}(\tilde{x},\tilde{\lambda}) - \tilde{h}(\tilde{x},\tilde{\lambda}) = 0, \tilde{\lambda} \in \tilde{\Lambda}, \tilde{x} \in \tilde{X} \tag{12}$$

Proof: We note that,

$$\tilde{g}(\tilde{x},\tilde{\lambda}) - \tilde{h}(\tilde{x},\tilde{\lambda}) = \widetilde{max}_{\tilde{w}\in\tilde{X}}\{\tilde{\lambda}(\tilde{w}-\tilde{x}) - {}^{1}/_{2}\,\|\tilde{w}-\tilde{x}\|^2\} \tag{13}$$

Since $\tilde{X}$ is a convex fuzzy set. Suppose that (8) holds for some $x \in X$ and some $\tilde{\lambda} \in \tilde{\Lambda}$, we have that;

$$0 \le \widetilde{max}_{\tilde{w}\in\tilde{X}}\{\tilde{\lambda}\tilde{w} - \tilde{\lambda}\tilde{x} - {}^{1}/_{2}\,\|\tilde{w}-\tilde{x}\|^2\} \le \widetilde{max}_{\tilde{w}\in\tilde{X}}\{\tilde{\lambda}\tilde{w} - \tilde{\lambda}\tilde{x} : \tilde{w} \in \tilde{X}\} = 0 \tag{14}$$

which yields $\tilde{g}(\tilde{x},\tilde{\lambda}) - \tilde{h}(\tilde{x},\tilde{\lambda}) = 0$. Suppose that $\tilde{g}(\tilde{x},\tilde{\lambda}) - \tilde{h}(\tilde{x},\tilde{\lambda}) = 0$ for some $\tilde{\lambda} \in \tilde{\Lambda}$, $\tilde{x} \in \tilde{X}$. Then we have that,

$$\widetilde{max}_{\tilde{w}\in\tilde{X}}\{\tilde{\lambda}(\tilde{w}-\tilde{x}) - {}^{1}/_{2}\,\|\tilde{w}-\tilde{x}\|^2\} = 0 \tag{15}$$

which implies that $\tilde{\lambda}(\tilde{w}-\tilde{x}) \le 0$ for all $\tilde{w} \in \tilde{X}$. In fact, if we have some $\tilde{w}_o \in \tilde{X}$ such that $\tilde{\lambda}(\tilde{w}_o - \tilde{x}) > 0$, then we can take a point $\overline{w}$ on line segment $[\tilde{w}_o, \tilde{x}]$ satisfying $\|\overline{w} - \tilde{x}\| < \|\tilde{\lambda}\| cos\theta$, where θ is the acute angel between $\tilde{\lambda}$ and $\tilde{w}_o - \tilde{x}$. Since $\tilde{X}$ is a fuzzy convex, then $\tilde{w} \in \tilde{X}$ but $\tilde{\lambda}(\tilde{w}-\tilde{x}) - {}^{1}/_{2}\,\|\tilde{w}-\tilde{x}\|^2 > 0$. It contradicts (15).

- **Note that:** The fuzzy functions $\tilde{g}$ and $\tilde{h}$ are fuzzy convex and differentiable.

From lemma 2.1, it follows that the problem can be formulated by the following of the difference convex fuzzy functions of differentiable programming with a fuzzy time-windows constraint:

$$\widetilde{min}\tilde{f}(\tilde{x})$$

subject to

$$\tilde{g}(\tilde{x},\tilde{\lambda}) - \tilde{h}(\tilde{x},\tilde{\lambda}) = 0, \forall\tilde{\lambda} \in \tilde{\Lambda}, \tilde{x} \in \tilde{X} \tag{16}$$

$$\tilde{t}_{v_i} + \tilde{t}_{v_{i,}v_j} \le \tilde{t}_{v_j}, \tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}], \tilde{t}_{v_j} \in \left[\tilde{a}_{v_j}, \tilde{b}_{v_j}\right], \tilde{t}_{v_i}, \tilde{t}_{v_{i,}v_j} \in \mathcal{R}^+, \forall v_i, v_j \in V$$

By Shigeno-Takahashi-Yamamoto [11], we see that the $\widetilde{\Lambda}$ in (16) could be replaced by $\{\tilde{\lambda}: \tilde{\lambda} \in \mathcal{R}_n^{**}, \tilde{\lambda} \ge, \tilde{\lambda}\tilde{e} = n^2\}$ where, $\mathcal{R}_n^{**} = \{\tilde{x}: \tilde{x} \in \mathcal{R}_n; \tilde{x} > 0\}$, $\mathcal{R}_n$ denotes the fuzzy set of n-dimensional real row vector and $\tilde{e}$ denoted to both the fuzzy row vector and the column fuzzy vector of ones. Then, we take the above set as $\widetilde{\Lambda}$ to design the algorithms.

III. A CONVEX FUZZY PROGRAMMING AND A CONVEX FUZZY ALGORITHM

A difference convex programming and a difference convex algorithm introduced by Pham Dinh Tao in 1985 and extensively developed in other works. A difference convex algorithm was successfully applied to a lot of the different and various nonconvex optimization problems to which it quite often gave a global solution and proved to be more robust and more efficient than related to standard methods, especially in the large-scale setting.

In [17] a difference convex algorithm is a primal-dual approach for finding a local optimum in the difference convex programming. More detailed the results see, [18] a difference convex algorithm can be found. Some numerical experiments are reported that it finds a global minimizer often if one chose a good start point.
Consider the following general problem:

$$\widetilde{w}_p = inf\{\tilde{g}(\tilde{x})\text{-}\ \tilde{h}(\tilde{x}): \tilde{x} \in \mathcal{R}^n\} \tag{14}$$

such that $\tilde{g}(.)$,$\tilde{h}(.)$: $\mathcal{R}^n \longrightarrow \mathcal{R} \cup \mathcal{R}$ are a low semi-continuous of the convex fuzzy functions on $\mathcal{R}^n$. It is easy to see that the problem;

$$\widetilde{min}\{\tilde{f}(\tilde{x})+\tilde{\delta}_{\tilde{X}}(\tilde{x})\text{-}\ \tilde{h}(\tilde{x})\}=\widetilde{min}\{\tilde{g}(\tilde{x})\text{-}\ \tilde{h}(\tilde{x})\} \tag{15}$$

where, $\tilde{\delta}_{\tilde{X}}$ the indicator of $\tilde{X}$ and $\tilde{g}(\tilde{x}) = \tilde{f}(\tilde{x}) + \tilde{\delta}_{\tilde{X}}(\tilde{x})$ is a special case of (14) as shown in (15) under the conservation of ∞. We also suppose that $\tilde{g}(\tilde{x})$- $\tilde{h}(\tilde{x})$ is bounded below on $\mathcal{R}^n$. The Ɛ-subgradient of $\tilde{g}$ at the point $\tilde{x}_0$ are defined by:

$$\partial_{\varepsilon}\tilde{g}(\tilde{x}_0) = \{\tilde{y} \in \mathcal{R}^n: \tilde{g}(\tilde{x}) \geq \tilde{g}(\tilde{x}_0) + \langle \tilde{x} - \tilde{x}_0, \tilde{y} \rangle - \varepsilon; \forall \tilde{x} \in \tilde{X}\} \tag{16}$$

and $\partial\tilde{g}(\tilde{x}_0) = \partial_0\tilde{g}(\tilde{x}_0)$. The conjugate fuzzy function of $\tilde{g}$ is given by:

$$\tilde{g}^*(\tilde{y}) = sup\{\langle \tilde{x} - \tilde{y} \rangle - \tilde{g}(\tilde{x}): \tilde{x} \in \mathcal{R}^n\} \tag{17}$$

From the low semi-continuous of $\tilde{g}$ and $\tilde{h}$, we see that $\tilde{g} = \tilde{g}^{**}$ and $\tilde{h} = \tilde{h}^{**}$ hold. Consider the dual fuzzy problem of (14):

$$\widetilde{w}_d = inf\{\tilde{h}^*(\tilde{y})\text{-}\ \tilde{g}^*(\tilde{y}): \tilde{y} \in \mathcal{R}^n\}$$

We have that;

$$\begin{aligned} \widetilde{w}_p &= inf\{\tilde{g}(\tilde{x})\text{-}\ \tilde{h}(\tilde{x}): \tilde{x} \in \mathcal{R}^n\} \\ &= inf\{\tilde{g}(\tilde{x}) \text{ - } sup\{\langle \tilde{x}, \tilde{y} \rangle - \tilde{h}^*(\tilde{y})\}: \tilde{y} \in \mathcal{R}^n: \tilde{x} \in \mathcal{R}^n\} \\ &= inf\{\tilde{g}(\tilde{x}) + inf\{\tilde{h}^*(\tilde{y}) - \langle \tilde{x}, \tilde{y} \rangle\}: \tilde{y} \in \mathcal{R}^n: \tilde{x} \in \mathcal{R}^n\} \\ &= inf\{\tilde{h}^*(\tilde{y}) + inf\{\tilde{h}^*(\tilde{y}) - \langle \tilde{x}, \tilde{y} \rangle\}: \tilde{y} \in \mathcal{R}^n\} \\ &= inf\{\tilde{h}^*(\tilde{y}) + sup\{\langle \tilde{x}, \tilde{y} \rangle - \tilde{g}(\tilde{x})\}: \tilde{x} \in \tilde{X}\}: \tilde{y} \in \mathcal{R}^n\} \\ &= inf\{\tilde{h}^*(\tilde{y}) \text{ - } \tilde{g}^*(\tilde{y}): \tilde{y} \in \mathcal{R}^n\} = \widetilde{w}_d \end{aligned} \tag{19}$$

For a pair $(\tilde{x},\tilde{y})$, Fenchel's inequality $\tilde{g}(\tilde{x})$-$\tilde{g}^*(\tilde{y}) \geq \langle\tilde{x},\tilde{y}\rangle$ holds for any proper of the convex fuzzy function $\tilde{g}$ and $\tilde{g}^*$. If $\tilde{y} \in \partial\tilde{g}(\tilde{x})$ then $\tilde{g}(\tilde{x}) + \tilde{g}^*(\tilde{y}) = \langle\tilde{x},\tilde{y}\rangle$.

Definition 3.1 A point $\tilde{x}^*$ is said to be a local fuzzy minimal of $\tilde{g} - \tilde{h}$ if there exists a neighborhood $\tilde{N}$ of $\tilde{x}^*$ such that $(\tilde{g} - \tilde{h})(\tilde{x}) \geq (\tilde{g} - \tilde{h})(\tilde{x}^*), \forall\, \tilde{x} \in \tilde{N}$.

Lemma 3.1 A point $\tilde{x}^*$ is a local fuzzy minimal for $\tilde{g} - \tilde{h}$, then $\partial\tilde{h}(\tilde{x}^*) \subseteq \partial\tilde{g}(\tilde{x}^*)$.

Proof: Let $(\tilde{g} - \tilde{h})(\tilde{x}) \geq (\tilde{g} - \tilde{h})(\tilde{x}^*), \forall\, \tilde{x} \in \tilde{N}$. Then $\tilde{g}(\tilde{x}) - \tilde{g}(\tilde{x}^*) \geq \tilde{h}(\tilde{x}) - \tilde{h}(\tilde{x}^*)$. Taking $\tilde{z} \in \partial\tilde{h}(\tilde{x}^*)$, we have $\tilde{h}(\tilde{x}) \geq \tilde{h}(\tilde{x}^*) + \langle\tilde{x} - \tilde{x}^*, \tilde{z}\rangle$ for all $\tilde{x} \in \mathcal{R}^n$. There for, we see that $\tilde{g}(\tilde{x}) \geq \tilde{g}(\tilde{x}^*) + \langle\tilde{x} - \tilde{x}^*, \tilde{z}\rangle$ for $\tilde{x} \in \tilde{N}$. We note that $\tilde{g}$ is a fuzzy convex, then $\tilde{g}(\tilde{x}) \geq \tilde{g}(\tilde{x}^*) + \langle\tilde{x} - \tilde{x}^*, \tilde{z}\rangle$ holds for $\tilde{x} \in \mathcal{R}^n$.

3.1 A Difference Convex Fuzzy Algorithm with a Fuzzy Time-Windows

We describe the framework of the difference fuzzy convex algorithm with a fuzzy time-windows by the first algorithm;

We describe the framework of the difference fuzzy convex algorithm with a fuzzy time-windows by the first algorithm;

step 0: pick up a fuzzy point $\tilde{x}^0 \in dom(\tilde{h})$, calculate $\tilde{y}^0 \in \partial\tilde{h}(\tilde{x}^0)$; $k = 1$;

step 1: each fuzzy point has satisfied a fuzzy time-windows constraint, i.e.,

$$\tilde{t}_{v_i} + \tilde{t}_{v_i,v_j} \leq \tilde{t}_{v_j}, \tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}], \tilde{t}_{v_j} \in [\tilde{a}_{v_j}, \tilde{b}_{v_j}], \tilde{t}_{v_i}, \tilde{t}_{v_i,v_j} \in \mathcal{R}^+, \forall v_i, v_j \in V;$$

step 2: calculate $\tilde{x}^k \in arg\widetilde{min}\{\tilde{g}(\tilde{x}) - (\tilde{h}(\tilde{x}^{k-1}) + \langle\tilde{x} - \tilde{x}^{k-1}, \tilde{y}^{k-1}\rangle) : \tilde{x} \in \mathcal{R}^n\}$;

calculate $\tilde{y}^k \in arg\widetilde{min}\{\tilde{h}(\tilde{y}) - (\tilde{g}^*(\tilde{y}^{k-1}) + \langle\tilde{x}^k, \tilde{y} - \tilde{y}^{k-1}\rangle) : \tilde{y} \in \mathcal{R}^n\}$;

step 3: if $\partial\tilde{h}(\tilde{x}^k) \cap \partial\tilde{g}(\tilde{x}^k) \neq \emptyset$, stop; otherwise, $k = k + 1$ go to step 1.

Lemma 3.1.1 Suppose that a fuzzy points $\tilde{x}^k$ and $\tilde{y}^k$ are satisfied a fuzzy time-windows constraint and generated in the above first algorithm, then $\tilde{x}^k \in \partial\tilde{h}^*(\tilde{y}^k)$ and $\tilde{y}^{k-1} \in \partial\tilde{g}(\tilde{x}^k)$.

Proof: Assume that $\tilde{x}^{k-1}$ and $\tilde{y}^{k-1}$ are satisfied a fuzzy time-windows constraint and in hand. We have,

$$\widetilde{min}\{\tilde{g}(\tilde{x}) - (\tilde{h}(\tilde{x}^{k-1}) + \langle\tilde{x} - \tilde{x}^{k-1}, \tilde{y}^{k-1}\rangle) : \tilde{x} \in \mathcal{R}^n\} \quad (20)$$

$$= \widetilde{min}\{\tilde{g}(\tilde{x}) - (\langle\tilde{x} - \tilde{y}^{k-1}\rangle) : \tilde{x} \in \mathcal{R}^n\} - (\tilde{h}(\tilde{x}^{k-1}) + (\langle\tilde{x} - \tilde{y}^{k-1}\rangle)\}$$

and $\widetilde{min}\{\tilde{h}^*(\tilde{y}) - (\tilde{g}^*(\tilde{y}^{k-1}) + \langle\tilde{x}^k, \tilde{y} - \tilde{y}^{k-1}\rangle) : \tilde{y} \in \mathcal{R}^n\} =$

$$= \widetilde{min}\{\tilde{h}^*(\tilde{y}) - (\langle\tilde{x}^k - \tilde{y}^k\rangle) : \tilde{y} \in \mathcal{R}^n\} - (\tilde{g}^*(\tilde{y}^{k-1}) + (\langle\tilde{x}^k - \tilde{y}^{k-1}\rangle)\}. \quad (21)$$

Thus, from step 2 in the above of the first algorithm,

$$\tilde{g}(\tilde{x}) - \langle\tilde{x}, \tilde{y}^{k-1}\rangle \geq \{(\tilde{g}(\tilde{x}^k)(\langle\tilde{x}^k - \tilde{y}^{k-1}\rangle)\}, \forall\, \tilde{x} \quad (22)$$

and $\quad h\ (y) - \langle x^k - y\rangle \geq \left(h\ (y) - \langle x^k - y^k\rangle\right)\}, \forall\, y$ (23)

It yields $\tilde{x}^k \in \partial\tilde{h}^*(\tilde{y}^k)$ and $\tilde{y}^{k-1} \in \partial\tilde{g}(\tilde{x}^k)$.

IV. METHOD AND ALGORITHM OF THE MMNFFPFTW

Let $\tilde{g}(\tilde{x})$, $\tilde{h}(\tilde{x})$ are the fuzzy convex functions, we can write the problem;

$$\widetilde{min}\{\tilde{f}(\tilde{x}) + \tilde{\delta}_X(\tilde{x}) - \tilde{h}(\tilde{x})\} = \widetilde{min}\{\tilde{g}(\tilde{x}) - \tilde{h}(\tilde{x})\} \quad (24)$$

In this section, we give an algorithm to solve the above problem (24). A general framework of the branch-and-bound algorithm with a fuzzy time-windows can be stated as follows.

- Algorithm General Framework

step 0: initial setting and calculating;

step 1: branching of the operation with a fuzzy time-windows constraint, i.e.,

$$\tilde{t}_{v_i} + \tilde{t}_{v_i,v_j} \leq \tilde{t}_{v_j}, \tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right], \tilde{t}_{v_j} \in \left[\tilde{a}_{v_j}, \tilde{b}_{v_j}\right], \tilde{t}_{v_i}, \tilde{t}_{v_i,v_j} \in \mathcal{R}^+, \forall v_i, v_j \in V;$$

step 2: local search for a smaller fuzzy upper bound;

step 3: find a larger fuzzy lower bound;

step 4: remove some regions, do to step 1.

We describe the steps 1, 2 and 3 as the following explained:

i) Describe step 1: Branching operation with a fuzzy time-windows constraint

A simplex based to the division is usually exploited in branch-and-bound method. At some steps, the contemporary simplex S is divided into two a smaller one S_1 and S_2. Branch-and-bound convergence of the algorithms and a fuzzy time-windows constraint, we need to division exhaustive, i.e., a nested sequence of the simplexes $\{S_k\}, k = 1,2, \ldots$ has the following properties:

1. $int(S_i) \cap int(S_j) = \emptyset; i \neq j;$ with satisfy a fuzzy time-windows constraints and $S_{k+1} \subseteq S_k; \forall k,$
2. $\lim_{k\to\infty} \cap_{k=1}^{\infty} S_k = \tilde{x}^0$ for some $\tilde{x}^0$.

At each step, we chose to divide the simplex S_k into two smaller ones S_{2k} and S_{2k+1} by bisecting a longest arc of S_k. The sequence $\{S_k\}, k = 1,2, \ldots$ in such process is exhaustive.

ii) Describe step 2: A local search for the smaller fuzzy upper bound

There are many methods to do a local search. Here, we exploit the first algorithm in this step. Even the first algorithm is not going to find a global optimum theoretically, but in many numerical experiments, it finds a global optimum practically.

As shown in problem (24) can be rewritten as a difference convex fuzzy programming $\min\{\tilde{g}-\tilde{h}\}$, then we can use the first algorithm to obtain a local fuzzy optimal solution. Then we assume that $\tilde{O}^i = \tilde{X} \cap S_i$ of the first algorithm which is a local fuzzy optimal solution satisfy the fuzzy time-windows constraint on $\tilde{X} \cap S_i$ by using the first algorithm.

iii) Describe step 3: Find a larger fuzzy lower bound

Assume that $\tilde{l}_{v_i}(\tilde{x})$ is the affine fuzzy function such that $\tilde{l}_{v_i}(v_j) = \tilde{h}(v_j), \forall v_j \in V(S_i)$ a fuzzy time-windows to be a non-negative time. From the convex fuzzy of $\tilde{h}(\tilde{x})$, we have $\tilde{l}_{v_i}(\tilde{x}) \geq \tilde{h}(\tilde{x}), \forall \tilde{x} \in S_i$, then,

$$\tilde{L}(\tilde{X} \cap S_i) = \widetilde{min}\{\tilde{f}(\tilde{x}) + \delta_{\tilde{X} \cap S_i}(\tilde{x}) - \tilde{l}_{v_i}(\tilde{x}) : \tilde{x} \in \mathcal{R}^n\}$$
$$\leq \widetilde{min}\{\tilde{f}(\tilde{x}) + \delta_{\tilde{X} \cap S_i}(\tilde{x}) - \tilde{h}(\tilde{x}) : \tilde{x} \in \mathcal{R}^n\}$$

Moreover, if $V(S_i) = \{v_1, \dots, v_{p_i}\}$ is a hand, satisfy a fuzzy time-windows constraint, then it is easy to calculate $\tilde{L}(\tilde{X} \cap S_i)$ because,

$$\widetilde{min}\{\tilde{f}(\tilde{x}) + \delta_{\tilde{X} \cap S_i}(\tilde{x}) - \tilde{l}_{v_i}(\tilde{x}) : \tilde{x} \in \mathcal{R}^n\} = \widetilde{min}\{\tilde{d} \textstyle\sum_{j=1}^{p_i} \tilde{\lambda}_j v_j +$$

$$\tilde{u} \textstyle\sum_{j=1}^{p_i} \tilde{\lambda}_j \tilde{r}(v_j) : \sum_j \tilde{\lambda}_j = 1, \tilde{\lambda}_j \geq 0, \tilde{A} \sum_{j=1}^{p_i} \tilde{\lambda}_j v_j = \tilde{b}, 0 \leq \sum_{j=1}^{p_i} \tilde{\lambda}_j v_j \leq \tilde{c}, \tilde{d} \in \mathcal{R}^n\} \quad (26)$$

where, $\tilde{\delta}$ are the sets of arcs which leaves and enters the source vertex, $\tilde{A}$ is a fuzzy matrix stands for the vertex arc incident a relationship in the fuzzy network, $\tilde{u}(\tilde{x}) \geq 0$ and $\tilde{r}(\tilde{x}) \geq 0, \forall \tilde{x} \in \tilde{X}$ is a concave function on $\tilde{X}$. Based on the above discussion, we give the following algorithm of the difference convex fuzzy algorithm of the MMNFFPFTW.

- A Difference Convex Algorithm of the MMNFFPFTW

step 0: let $\tilde{\varepsilon}$ and S_0 such that $\tilde{X} \subseteq S_0$. Let $\tilde{x}_0 = 0, \tilde{x}_0 = (-1, \dots, -1)$, $\tilde{b}^U \geq 0$, $\tilde{b}_L =$ $\widetilde{min}\{\tilde{f}(\tilde{x}) : \tilde{A}\tilde{x} = \tilde{b}, 0 \leq \tilde{x} \leq \tilde{b}\}, \widetilde{M} = S_0$;

step 1: select $S_0 \in \widetilde{M}$ such that $\tilde{b}_{\tilde{L}} = \tilde{L}(\tilde{X} \cap S_0)$ and dived S_0 into S_1 and S_2;

step 2: $\tilde{O}^i = \tilde{X} \cap S_i$ from the first algorithm for all $i = 1,2$ if $\tilde{O}^i < \tilde{b}^U$ then $\tilde{O}^i = \tilde{b}^U$;

step 3: if $\tilde{b}_L < L(\tilde{X} \cap S_0)$ then $\tilde{b}_L = \tilde{L}(\tilde{X} \cap S_0)$, if $\tilde{b}^U - \tilde{b}^U < \tilde{\varepsilon}$ then Stop;

step 4: $\widetilde{M} = \{S \in \widetilde{M} : \tilde{L}(\tilde{X} \cap S_0) < \tilde{b}^U\}$, if $\widetilde{M} = \emptyset$ then Stop, otherwise, go to step 1.

The convergence of the above algorithm of the MMNFFPFTW is exhaustive of the partition.

V. CONCLUSION

This paper presents a new version of the Minimum Maximal Network Flow Problem (MMNFP), a new version is the Minimum Maximal Network Fuzzy Flow Problem with Fuzzy Time-Windows (MMNFFPFTW). We consider a generalized fuzzy version of the minimum maximal network flow problem in fuzzy networks. We propose a mathematical model with a dual formulation of the MMNFFPFTW. Also, we propose a new algorithm of the MMNFFPFTW. Our algorithm is a class of a branch-and-bound, the result achieved in this paper to illustrates the promising application prospects for the algorithm using the fuzzy networks model.

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