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I. INTRODUCTION

A Mersenne number M_x is a number that is one less than a power of two

$$M_x = 2^x - 1.$$

A Mersenne prime is a Mersenne number that is prime.

The first few Mersenne primes are:

3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951, 618970019642690137449562111,... (A000668 in the OEIS).

The first few exponents x that give Mersenne primes are:

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, ... (A000043 in the OEIS).

Mersenne primes are named after Marin Mersenne, a French Minim friar who studied them in the early 17th century.

Many great mathematicians studied Mersenne primes and left many interesting stories. We do not repeat those stories [13].

Euclid proved that if $2^p - 1$ is prime, then $2^{p-1}\ (2^p - 1)$ is an even perfect number. Euler proved that, conversely, all even perfect numbers have this form [1]. So that the search for Mersenne primes is also the search for even perfect numbers. This is well known as the Euclid–Euler theorem.

On December 21, 2018, it was announced that The Great Internet Mersenne Prime Search (GIMPS) discovered the largest known prime number, the 51-th Mersenne prime, $2^{282589933} - 1$, having 24862048 digits. A comput-

[1]2010 Mathematics Subject Classi cation . Primary 11N35; Secondary 11N45, 11A07,11Y16, 11B37,11B50.

er volunteered by Patrick Laroche from Ocala, Florida, made the find on December 7, 2018.[4].

Mathematicians believe that the set of Mersenne primes is infinite. "But we are still missing a proof that this guess is true. We are still waiting for a modern day Euclid to prove that Mersenne's primes never run dry. Or perhaps this far-off peak is just a mathematical mirage." in "The music of the primes" the mathematician Marcus du Sautoy said[13].

Lenstra--Pomerance--Wagstaff conjectured that there are infinitely many Mersenne primes [2], [3], and made a quantitative variant formula based on the heuristics model of primes. The number of Mersenne primes up to x is

$$e^{\gamma} \times \log\log(x)/\log(2).$$

It is also not known whether infinitely many Mersenne numbers with prime exponents are composite.

The author had published that there are infinitely many Mersenne composite numbers with prime exponents by the recursive sieve method [9] [10]. In this paper, we prove that there are infinitely many Mersenne primes by an interaction between proof and algorithm, like the proof of Chinese remainder theorem.

In section 2, we repeat a sifting process of primes and obtain the recursive formula for primes p_i. By slightly refining the sifting process of primes, we design a recursive sieve for Mersenne primes.

In section 3, we prove Mersenne prime conjecture based on order topological theory for the sifting process.

II. A RECURSIVE SIEVE METHOD FOR MERSENNE PRIMES

Within the framework of recursion theory, we reformulate Eratosthenes sieve method and invent recursive sieve method, which is a modulo algorithm on sets of natural numbers

For expressing this modulo algorithm by well formed formulas, we need to extend both basic operations addition and multiplication $+$, $\times$ into finite sets of natural numbers, and introduce several definitions and notation.

We use small letters a, x, t to denote natural numbers and capital letters A, X, T to denote sets of natural numbers except for M_x.

For arbitrary both finite sets of natural numbers A, B we write

$$A = \langle a_1, a_2, \ldots, a_i, \ldots, a_n \rangle,\ a_1 < a_2 < \cdots < a_i < \cdots < a_n,$$

$$B = \langle b_1, b_2, \ldots, b_j, \ldots,\ b_m \rangle,\ b_1 < b_2 < \cdots < b_j < \cdots < b_m.$$

We define

$$A + B = \langle a_1 + b_1, a_2 + b_1, \ldots, a_i + b_j \ldots, a_{n-1} + b_m, a_n + b_m \rangle,$$
$$AB = \langle a_1 b_1, a_2 b_1, \ldots, a_i b_j \ldots, a_{n-1} b_m, a_n b_m \rangle.$$

Example:

$$\langle 1, 5 \rangle + \langle 0, 6, 12, 18, 24 \rangle = \langle 1, 5, 7, 11, 13, 17, 19, 23, 25, 29 \rangle,$$
$$\langle 6 \rangle \langle 0, 1, 2, 3, 4 \rangle = \langle 0, 6, 12, 18, 24 \rangle.$$

For the empty set $\emptyset$ we define $\emptyset + B = \emptyset$ and $\emptyset B = \emptyset$.
We write $A \setminus B$ for the set difference of A and B.
Let

$$X \equiv A = \langle a_1, a_2, \ldots, a_i, \ldots, a_n \rangle \mod a$$

be several residue classes $\mod a$.
If $\gcd(a, b) = 1$, we define the solution of the system of congruences

$$X \equiv A = \langle a_1, a_2, \ldots, a_i, \ldots, a_n \rangle \mod a,$$
$$X \equiv B = \langle b_1, b_2, \ldots, b_j, \ldots, b_m \rangle \mod b$$

to be

$$X \equiv D = \langle d_{11}, d_{21}, \ldots, d_{ij}, \ldots, d_{n-1m}, d_{nm} \rangle \mod ab,$$

where $x \equiv d_{ij} \mod ab$ is the solution of the system of congruences

$$x \equiv a_i \mod a,$$
$$x \equiv b_j \mod b.$$

The solution $X \equiv D \mod ab$ is computable and unique by the Chinese remainder theorem.

For example, $X \equiv D = \langle 5, 25 \rangle \mod 30$ is the solution of the system of congruences

$$X \equiv \langle 1, 5 \rangle \mod 6,$$
$$X \equiv \langle 0 \rangle \mod 5.$$

We known that the residue class $a_i \mod a$ is the set of natural numbers $\{x : x \equiv a_i \mod a\}$, several residue classes $A \mod a$ is the union of several sets. Thus we may write the relation $x \in A \mod a$ and set operation $B \cup (A \mod a)$.

Except extending $+, \times$ into finite sets of natural numbers, we continue the traditional interpretation of the formal language $0, 1, +, \times, \in$. The reader who is familiar with model theory may know, we have founded a new model or structure of a second-order arithmetic by a two-sorted logic

$$\langle P(N), N, +, \times, 0, 1, \in \rangle,$$

and a formal system

$$\langle P(N), N, +, \times, 0, 1, \in\rangle \models PA \cup ZF,$$

where N is the set of all natural numbers, and $P(N)$ is the power set of N; PA is the Peano theory, and ZF is the set theory.

We denote this model by $P(N)$.

The second-order language $\langle 0, 1, +, \times, \in\rangle$ has stronger expressive power. The second-order formal system $P(N)$ has some exotic mathematical structures in terms of sets of natural numbers; the first order formal system has no such structures.

Traditional recursion theory discusses functions, their inputs and outputs are natural numbers. We have computed out 51 Mersenne primes, but the evidence does not provide theoretical information about infinitude.

In the second-order arithmetics $P(N)$, we may construct recursive functions on sets of natural numbers by arithmetical operations $+, \times$ and set-theoretical operations $\cup, \cap, \setminus$, their inputs and outputs are sets of natural numbers. Then we obtain a sequence of sets of natural numbers (T_i'), which converges to the set T_e of all exponents of Mersenne primes. We reveal an exotic structure of the set T_e. The existing theory of those structures, order topology, allows us to prove the conjecture.

Now we repeat a sifting process for primes.
Let p_i be the i-th prime, $p_0 = 2$. Let

$$m_{i+1} = \prod_{0}^{i} p_j.$$

From the entire set of natural numbers, we successively delete the residue class $0 \mod p_0, 0 \mod p_1, \ldots, 0 \mod p_i$, i.e., the set of all numbers x such that the least prime factor of x is p_i, instead of the multiples of p_i in a given range. Then the left residue class $T_{i+1} \mod m_{i+1}$ is the set of all numbers x such that x does not contain any prime $p_j \le p_i$ as a factor $(x, m_{i+1}) = 1$.

Let T_{i+1} be the set of least nonnegative representatives of the left residue class $T_{i+1} \mod m_{i+1}$. Then we obtain a recursive formula for the set T_{i+1} and the prime p_{i+1}, which represents the recursive sieve method for primes [9].

$$\begin{aligned}
T_1 &= \langle 1\rangle, \\
p_1 &= 3, \\
T_{i+1} &= (T_i + \langle m_i\rangle\langle 0, 1, 2, \ldots, p_i - 1\rangle) \setminus \langle p_i\rangle T_i, \\
p_{i+1} &= g(T_{i+1}),
\end{aligned} \tag{2.1}$$

where $X \equiv \langle p_i \rangle T_i \mod m_{i+1}$ is the solution of the system of congruences

$$X \equiv T_i \; mod \, m_i,$$

$$X \equiv \langle 0 \rangle \; mod \, p_i,$$

and $g(T)$ is a projective function, which takes the smallest number in the set but the number 1.

$$g(T) = g(\langle t_1, t_2, \ldots, t_n \rangle) = t_2.$$

The cardinality of the set T_{i+1} is

$$|T_{i+1}| = \prod_{0}^{i} (p_j - 1). \tag{2.2}$$

We exhibit the first few terms of formula (2.1)

$$\begin{aligned}
T_1 &= \langle 1 \rangle, \\
p_1 &= 3, \\
T_2 &= (\langle 1 \rangle + \langle 2 \rangle \langle 0, 1, 2 \rangle) \setminus \langle 3 \rangle = \langle 1, 5 \rangle, \\
p_2 &= 5, \\
T_3 &= (\langle 1, 5 \rangle + \langle 6 \rangle \langle 0, 1, 2, 3, 4 \rangle) \setminus \langle 5, 25 \rangle = \langle 1, 7, 11, 13, 17, 19, 23, 29 \rangle, \\
p_3 &= 7.
\end{aligned}$$

It is easy to prove this primitive recursive formula by mathematical induction.

In contrast with Eratosthenes sieve, which does not automatically provide theoretical information, the recursive sieve method itself mechanically provides a constructive proof of Euclid's theorem, that there are infinitely many primes[9].

This formula gives a recursive structure for every prime. Thus primes do not appear randomly; they are computed one after the other by $+, \times$. They are governed by a recursive rule. Recursion opens new theoretical windows onto our understanding of the primes. It seems this is a prime conspiracy.

We may directly extend this constructive proof into several arithmetic progressions, for example, $3x - 1$ or $4x - 1$ or $6x - 1$. We can not extend this constructive proof into Mersenne primes.

We rewrite the sifting process for primes to design a sieve for Mersenne primes and prove the Mersenne prime conjecture.

Based on the recursive sieve method for primes, formula (2.1), we successively delete all numbers x such that x contains the least prime factor p_i, we delete all composites together with the prime p_i. The sifting condition or 'sieve' is

$$x \equiv 0 \; mod \, p_i \wedge p_i \leq x.$$

We modify the sifting condition to be

$$x \equiv 0 \; mod \, p_i \wedge p_i < x. \tag{2.3}$$

With this new sifting condition or 'sieve' we successively delete the set C_i of all numbers x such that x is composite with the least prime factor p_i,

$$C_i = \{x : x \in T_i \mod m_i \wedge x \equiv 0 \mod p_i \wedge p_i < x\},$$

but save the prime p_i.

We delete all composite sets C_j with $0 \leq j < i$ from the set N of all natural numbers and leave a sifted set

$$L_i = N \setminus \bigcup_{0}^{i-1} C_j.$$

In the sifted set, every number x does not contain any prime p_j, with $0 \leq j < i$, as factor except itself.

We delete all composite sets C_i and leave the end-sifted set, which is set of all primes T_e,

$$T_e = N \setminus \bigcup_{0}^{\infty} C_i.$$

Let A_i be the set of all primes less than p_i, the set of survivors

$$A_i = \langle 2, 3, 5, 7, \ldots, p_{i-1} \rangle.$$

From the recursive formula (2.1), we deduce that the sifted set L_i is the union of the set A_i of survivors and the residue class $T_i \mod m_i$,

$$L_i = A_i \cup (T_i \mod m_i) \tag{2.4}$$

Now we modify the sifting condition (2.3) to obtain sets of exponents of Mersenne primes.

From the identity

$$2^{ab} - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + 2^{3a} + \cdots + 2^{(b-1)a}),$$

we know that the Mersenne sequence is a divisibility sequence. In other words, M_n divides M_m if and only if n divides m. It follows that every Mersenne prime M_p must has a prime exponent, but not every Mersenne number M_p with a prime exponent is Mersenne prime; any both Mersenne numbers M_p and M_q with prime exponents are coprime.

We discuss the set T_e of all exponents of Mersenne primes and its infinitude in the set N of all exponents of Mersenne numbers.

If we only consider a Mersenne number as a divisor of Mersenne numbers inside the divisibility system of Mersenne numbers, then the set T_e of all primes is the set of all exponents of Mersenne primes, every Mersenne number M_p with prime exponent does not contain any Mersenne number M_x as a factor except 1 and itself.

A Mersenne number M_q with prime exponent may contain normal prime factors, which are not numbers of the form $2^x - 1$, for example, $M_{11} = 23 \times 89$. Thus we must remove every prime q in the set T_e of all primes if M_q contains a normal prime p as a factor except itself.

Note the notion of divisibility inside the system of Mersenne numbers is different from the usual notion.

About normal prime factors of a Mersenne number with prime exponent, we have known some simple facts [1].

(1) A prime number divides at most one Mersenne number with prime exponent.
(2) Let p be a prime. Then there is a number x such that $p|M_x$ if only if there is a number $c < p$ such that $p|M_c$.
(3) If q is an odd prime, then every prime p that divides M_q is congruent to $\pm 1 \mod 8$.
(4) If q is an odd prime, then every prime p that divides M_q must be 1 plus a multiple of $2q$, $p - 1 = 2kq$. This holds even when M_q is prime.
(5) Let $p \equiv 3 \mod 4$ be prime and $2p + 1$ is also prime, if and only if $2p + 1$ divides M_p.

One proves (1) by coprime. One proves (2),(3),(4),(5) by Fermat's little theorem

$$a^{p-1} \equiv 1 \mod p.$$

It follows that we only need to modify successively the set A_i of survivors for each prime p_i to obtain a new set A_{i+1} of survivors, such that if $a \in A_{i+1}$, then M_a contains neither normal prime $p \leq p_i$ nor Mersenne prime $M_p \leq M_{p_i}$ as a factor except itself.

Given any prime p_i, suppose that we have a modified set A_i, then we obtain the next set A_{i+1} by the following rules.

If the prime p_i is congruent to $\pm 1 \mod 8$, let $p_i - 1 = 2kq$, and if there is an odd prime q in the set A_i such that

$$p_i < M_q \wedge p_i | M_q,$$

then M_q is a Mersenne composite, which contains the least normal prime factor p_i. We remove the prime q from the set A_i and add the prime p_i to obtain the set A_{i+1}.

$$A_{i+1} = (A_i \cup \langle p_i \rangle) \setminus \langle q \rangle.$$

If there is no such a prime q, example $p_i \equiv 3, 5 \mod 8$, then for every number x in the sifted set L_i the Mersenne number M_x does not contain the normal prime p_i as a factor. We add the prime p_i into the set A_i

$$A_{i+1} = A_i \cup \langle p_i \rangle.$$

Now for every number x in the sifted set L_{i+1}, the Mersenne number M_x does not contain Mersenne number M_{p_i} as a factor and does not the normal prime p_i as a factor also except itself.

We exhibit the first few terms of modified sets A_i as examples.

$$\begin{aligned}
A_1 &= \langle 2 \rangle, \\
A_2 &= \langle 2, 3 \rangle, \\
A_3 &= \langle 2, 3, 5 \rangle, \\
A_4 &= \langle 2, 3, 5, 7 \rangle, \\
A_5 &= \langle 2, 3, 5, 7, 11 \rangle, \\
A_6 &= \langle 2, 3, 5, 7, 11, 13 \rangle, \\
A_7 &= \langle 2, 3, 5, 7, 11, 13, 17 \rangle, \\
A_8 &= \langle 2, 3, 5, 7, 11, 13, 17, 19 \rangle, \\
A_9 &= \langle 2, 3, 5, 7, 13, 17, 19, 23 \rangle, M_{11} = 23 \times 89, \\
A_{10} &= \langle 2, 3, 5, 7, 13, 17, 19, 23, 29 \rangle, \\
A_{11} &= \langle 2, 3, 5, 7, 13, 17, 19, 23, 29, 31 \rangle,
\end{aligned}$$

The sifting condition formula (2.3) is converted into

$$(x \equiv 0 \mod p_i \wedge p_i < x) \vee (M_x \equiv 0 \mod p_i \wedge p_i < M_x). \quad (2.5)$$

The recursive sieve (2.5) is a perfect tool; with this tool, we may delete exponents of all non-Mersenne primes and leave exponents of all Mersenne primes. So that we only need to determine the number of all Mersenne primes $|T_e|$. If we do so successfully, then the parity obstruction, a ghost in a house of primes, has been automatically evaporated.

With the recursive sieve (2.5), each exponent of non Mersenne prime is deleted exactly once; there is need neither the inclusion-exclusion principle nor the estimation of error terms, which cause all the difficulty in normal sieve theory.

According to this sifting condition or 'sieve' we successively delete the set C_i of all numbers x, such that M_x is non-Mersenne prime with the least factor of Mersenne number M_{p_i} except itself or with the least normal prime factor $p_i > 2$ except itself,

$$C_i = \{x : x \in (A_i \cup (T_i \mod m_i)) \wedge ((x \equiv 0 \mod p_i \wedge p_i < x) \vee (M_x \equiv 0 \mod p_i \wedge p_i < M_x))\}.$$

but remain the survivor x if $p_i = x$ or $p_i = M_x$.

We delete all sets of exponents of non Mersenne numbers C_j with $0 \leq j < i$ from the set N of all exponents of Mersenne numbers and leave a sifted set

$$L_i = N \setminus \bigcup_{0}^{i-1} C_j. \tag{2.6}$$

In the above sifted set, for every number x, Mersenne number M_x contains neither normal prime $p < p_i$ nor Mersenne prime $M_p < M_{p_i}$ as a factor except itself.

We delete all sets of exponents of non-Mersenne numbers C_j and leave the end sifted set T_e, which is the set of all exponents of Mersenne primes

$$T_e = N \setminus \bigcup_{0}^{\infty} C_i.$$

The set A_i of survivors is a set of exponents of Mersenne primes or almost Mersenne primes, the candidates. If $a \in A_i$ and $M_a < p_i^2$, then M_a is a Mersenne prime.

Obviously, we have the relation

$$|A_i| \leq |A_{i+1}|.$$

From the recursive formula (2.1), we deduce again that the sifted set L_i is the union of the set A_i of survivors and the residue class $T_i \mod m_i$.

$$L_i = A_i \bigcup (T_i \mod m_i). \tag{2.7}$$

Now we intercept an initial segment T_i' from the above sifted set L_i, which is the union of the set A_i of survivor and the set T_i of least nonnegative representatives, then we obtain a new recursive formula

$$T_i' = A_i \bigcup T_i. \tag{2.8}$$

Except remaining all survivor x less than p_i in the initial segment T_i', both sets T_i' and T_i are the same.

For example

$$\begin{aligned} A_3 &= \langle 2,3,5\rangle. \\ T_3' &= \langle 2,3,5\rangle \cup \langle 1,7,11,13,17,19,23,29\rangle \\ &= \langle 2,3,5,1,7,11,13,17,19,23,29\rangle. \end{aligned}$$

Formula (2.8) expresses a recursively sifting process according to the sifting condition (2.5) and provides a recursive definition of the initial segment T_i'.

The following readers will see that the initial segment is a well-chosen notation, which makes mathematical reasoning itself easier or even purely mechanical.

We consider some properties of the initial segment $T_i^{'}$, and the limit of the sequence of the initial segments (T_i') to determine the set of all exponents of Mersenne primes and its cardinality.

The number of elements of the initial segment T_i' is

$$|T_i'| = |A_i| + |T_i|. \tag{2.9}$$

From formula (2.2) we deduce that the cardinal sequence $(|T_i'|)$ is strictly increasing

$$|T_i'| < |T_{i+1}'|.$$

Based on cardinal arithmetics we have

$$\lim T_i' = \bigcup |T_i'| = \aleph_0.$$

Based on order topology, obviously, we have also

$$\lim |T_i'| = \aleph_0. \tag{2.10}$$

Intuitively we see that the initial segment T_i' approaches the end-sifted set T_e, and the corresponding cardinality $|T_i'|$ approaches infinity as $i \to \infty$. Thus the end-sifted set, the set of all exponents of Mersenne primes is limit computable and is an infinite set.

We give a formal proof.

III. THE INFINITUDE OF MERSENNE PRIMES

Let A_i' be the subset of all exponents of Mersenne primes in the initial segment T_i',

$$A_i' = \{x \in T_i' : x \text{ is an exponent of Mersenne prime}\}. \tag{3.1}$$

Example,

$$\begin{aligned}
A_1' &= \langle 2 \rangle, \\
A_2' &= \langle 2, 3, 5 \rangle, \\
A_3' &= \langle 2, 3, 5, 7, 13, 17, 19 \rangle, \\
A_4' &= \langle 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127 \rangle, \\
A_5' &= \langle 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281 \rangle,
\end{aligned}$$

We consider the properties of both sequences of sets (T_i') and (A_i') to prove Mersenne prime conjecture.

Theorem 3.1. *The sequence of the initial segments (T_i') and the sequence of its subsets (A_i') of exponents of Mersenne primes both converge to the set of all exponents of Mersenne primes T_e.*

First from set theory [5], next from order topology [8], we prove this theorem.

Proof. For convenience of the reader, we quote a definition of the set-theoretic limit of a sequence of sets [5].

Let (F_n) be a sequence of sets; we define $\limsup_{n=\infty} F_n$ and $\liminf_{n=\infty} F_n$ as follows.

$$\begin{aligned}
\limsup_{n=\infty} F_n &= \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{\infty} F_{n+i}, \\
\liminf_{n=\infty} F_n &= \bigcup_{n=0}^{\infty} \bigcap_{i=0}^{\infty} F_{n+i}.
\end{aligned}$$

It is easy to check that $\limsup_{n=\infty} F_n$ is the set of those elements x, which belongs to F_n for infinitely many n. Analogously, x belongs to $\liminf_{n=\infty} F_n$ if and only if it belongs to F_n for almost all n, that is, it belongs to all but a finite number of the F_n.

If

$$\limsup_{n=\infty} F_n = \liminf_{n=\infty} F_n,$$

we say that the sequence of sets (F_n) converges to the limit

$$\lim F_n = \limsup_{n=\infty} F_n = \liminf_{n=\infty} F_n.$$

From formula (2.6) we know that the sequence of sifted sets (L_i) is descending

$$L_1 \supset L_2 \supset \cdots \supset L_i \supset \cdots\cdots .$$

According to the definition of the set-theoretic limit of a sequence of sets, we obtain that the sequence of sifted sets (L_i) converges to the set T_e

$$\lim L_i = \bigcap L_i = T_e.$$

From definition (3.1) the sequence of subsets (A'_i) is ascending

$$A'_1 \subset A'_2 \subset \cdots \subset A'_i \subset \cdots\cdots ,$$

we obtain that the sequence of subsets (A'_i) converges to the set T_e,

$$\lim A'_i = \bigcup A'_i = T_e.$$

The initial segment T'_i locates between two sets A'_i and L_i

$$A'_i \subset T'_i \subset L_i.$$

It is easy to prove that the sequence of initial segments (T'_i) converges to the set T_e

$$\lim T'_i = T_e.$$

In general, for any sequence of finite sets (G_i), if G_i locates between two sets A'_i and L_i,

$$A'_i \subset G_i \subset L_i,$$

then we have

$$\limsup G_i \subset \lim L_i,$$

$$\liminf G_i \supset \lim A'_i.$$

Thus

$$\lim G_i = T_e,$$

According to set theory, we have proved that both sequences of sets (T_i') and (A_i') converge to the set of all exponents of Mersenne primes T_e.

$$\lim T_i' = \lim A_i' = T_e.$$

Even $T_e = \emptyset$ the limit of set theory is valid too.

We can not use analytic techniques for limits of set theory, so that we try to endow them with an order topology and prove that according to order topology, both sequences of sets (T_i') and (A_i') converge to the set of all exponents of Mersenne primes T_e.

We quote J.R.Munkres's definition of the order topology [7][8].

Let X be a set with a linear order relation; assume X has more one element. Let $\mathbb{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) in X.

(3) All intervals of the form $[a, b_0)$, where b_0 is the largest element (if any) in X.

The collection $\mathbb{B}$ is a bases of a topology on X, which is called the order topology.

The empty or singleton is not a linear order $<$ set. There is no order topology on the empty set or sets with a single element.

The recursively sifting process formula (2.8) produces both sequences of sets together with the common set-theoretic limit point T_e.

$$\mathbf{X_1}: \quad T_1', T_2', \ldots, T_i', \ldots\ldots; T_e,$$
$$\mathbf{X_2}: \quad A_1', A_2', \ldots, A_i', \ldots\ldots; T_e.$$

We further consider structures of both sets $\mathbf{X_1}$ and $\mathbf{X_2}$ using the recursively sifting process (2.8) as an order relation

$$i < j \to T_i' < T_j', \ \forall i(T_i' < T_e),$$
$$i < j \to A_i' < A_j', \ \forall i(A_i' < T_e).$$

The set $\mathbf{X_1}$ has no repetitious term. It is a well ordered set with the order type $\omega + 1$ using the recursively sifting process (2.8) as an order relation. Thus the set $\mathbf{X_1}$ may be endowed an order topology.

In general, the set $\mathbf{X_2}$ may have no repetitious term or may have some repetitious terms or may be a set with a single element $\mathbf{X_2} = \{\emptyset\}$; in other words, $A_i' = \emptyset$ for all i.

We have computed out 51 patterns of the first few exponents of Mersenne primes. The set $\mathbf{X_2}$ contains more than one element, it may be endowed an order topology using the recursively sifting process (2.8) as an order relation.

Obviously, for every neighborhood $(c, T_e]$ of T_e, there is a natural number i_0, for all $i > i_0$, we have $T_i' \in (c, T_e]$ and $A_i' \in (c, T_e]$. Thus both sequences of sets (T_i') and (A_i') converge to the set of all exponents of Mersenne primes T_e.

$$\begin{aligned} \lim A_i' &= T_e, \\ \lim T_i' &= T_e. \end{aligned}$$

According to order topology, we have again proved that both sequences of sets (T_i') and (A_i') converge to the set of all exponents of Mersenne primes T_e. We also have

$$\lim\ T_i' = \lim A_i'. \tag{3.2}$$

Only if $T_e = \emptyset$ under some sifting conditions, the set $\mathbf{X_2}$ only has a single element $\emptyset$, which has no order topology by definition. In this case, formula (3.2) is not valid, and we prove nothing by order topology.

In the last section, we shall discuss the existence of the order topological limits $\lim T_i', \lim A_i'$.

Theorem 3.1 and formula (2.10) reveal some particular order topological structures of the set of all exponents of Mersenne primes T_e on the sets $\mathbf{X_1}, \mathbf{X_2}$. Now we can prove that the set of all exponents of Mersenne primes is infinite set based on usual theorems of general topology.

Theorem 3.2. *The set of all exponents of Mersenne primes is infinite.*

We give two proofs.

Proof. A

We consider cardinalities $|T_i'|$ and $|A_i'|$ of sets on two sides of the equality (3.2), and order topological limits of cardinal sequences $(|T_i'|)$ and $(|A_i'|)$ with the usual order relation $\leq$, as both sets T_i' and A_i' tend to T_e.

From general topology, we know that if the limits of both cardinal sequences $(|T_i'|)$ and $(|A_i'|)$ on two sides of the equality (3.2) exist, then both limits are equal; if $\lim |A_i'|$ does not exist, then the condition for the existence of the limit $\lim |T_i'|$ is not sufficient [6].

For exponents of Mersenne primes, the set T_e is nonempty $T_e \neq \emptyset$, the formula (3.2) is valid. Obviously, the order topological limits $\lim |A'_i|$ and $\lim |T'_i|$ on two sides of the equality (3.2) exist, thus both limits are equal

$$\lim |A'_i| = \lim |T'_i|.$$

From formula (2.10) $\lim |T'_i| = \aleph_0$ we have

$$\lim |A'_i| = \aleph_0. \tag{3.3}$$

Usually, let $\pi(n)$ be the counting function, the number of exponents of Mersenne primes less than n. Normal sieve theory, analytic number theory, is unable to provide non-trivial lower bounds of $\pi(n)$ due to the parity problem. Let n be a natural number. Then the number sequence (m_i) is a subsequence of the number sequence (n), we obtain

$$\lim \pi(n) = \lim \pi(m_i).$$

By formula (3.1), the A'_i is the set of exponents of all Mersenne primes less than m_i, and the $|A'_i|$ is the number of exponents of all Mersenne primes less than m_i, thus $\pi(m_i) = |A'_i|$. We have

$$\lim \pi(m_i) = \lim |A'_i|.$$

$$\lim \pi(n) = \aleph_0. \tag{3.4}$$

We directly proved the conjecture with the counting function $\pi(n)$.

Next, we give another proof by the continuity of the cardinal function in the above particular order topological space.

The continuous function is a morphism between topological spaces, which preserves the topological structures.

The continuity depends only on the topologies of its domain and range spaces.

Proof. B
Let $f : \mathbf{X} \to \mathbf{Y}$ be the cardinal function from the order topological space $\mathbf{X}$ to the order topological space $\mathbf{Y}$, such that $f(T) = |T|$.

$$\begin{aligned} \mathbf{X} &: \quad T'_1,\ T'_2,\ \ldots,\ T'_i, \ldots\ldots;\ T_e, \\ \mathbf{Y} &: \quad |T'_1|, |T'_2|, \ldots, |T'_i|, \ldots\ldots; \aleph_0. \end{aligned}$$

It is easy to check that for every open set $[\ |T_1'|, |d|), (|c|, |d|), (|c|, \aleph_0]$ in $\mathbf{Y}$ the preimage $[T_1', d), (c, d), (c, T_e]$ is also an open set in $\mathbf{X}$. According to the definition of continuity of a function, the cardinal function $|T|$ is continuous at T_e with respect to the above order topology.

$$|\lim T_i'| = \lim |T_i'|. \tag{3.5}$$

The order topological spaces are Hausdorff spaces. In Hausdorff spaces, the limit point of the sequence of sets (T_i') and the limit point of the cardinal sequence $(|T_i'|)$ are unique if both exist.

We have proved theorem 3.1, $\lim T_i' = T_e$, and formula (2.10), $\lim |T_i'| = \aleph_0$. Substitute both into formula (3.5); we obtain that the set of exponents of all Mersenne primes is infinite,

$$|T_e| = \aleph_0. \tag{3.6}$$

In the last section, we discuss the existence of the limits $\lim |T_i'|$ and $\lim T_i'$.

Without any estimation or statistical data, without the Riemann hypothesis, by the recursive sieve method, we reveal the recursive structure, set theoretic structure and order topological structure of the set T_e on sequences of sets. The well known theories of those structures allow us to prove the Mersnne prime conjecture.

In the first order formal system there is no such proof.
We proved that the set of exponents all of Mersenne primes is an infinite set. In other words we have proved that the Mersenne prime conjecture is true.

Theorem 3.3. *There are infinitely many Mersenne primes.*
By the Euclid–Euler theorem we also prove

Theorem 3.4. *There are infinitely many even perfect numbers.*

Similarly, we may prove the Fibonacci prime conjecture, in another paper, we discuss this conjecture.

IV. DISCUSSION

In general, we can not prove that the cardinal function on sequences of sets is continuous. . There is a counterexample, the Ross-Littwood paradox [12] [14].

For example: consider the limit of the sequence of sets, which have no pattern

$$T_i = \langle i+1, i+2, \ldots, 10i \rangle.$$

From set theory we know $\lim T_i = T_e = \emptyset$, thus $|T_e| = 0$. But we also have $\lim |T_i| = \lim 10i = \infty$ from real analysis. If the cardinal function is continuous, then there is a contradiction in real analysis, the empty has an infinite cardinality. In this case, one can only get up the continuity and says that there is no relation between the cardinality of the end sifted set $|T_e| = 0$ and the limit $\lim |T_i| = \infty$.

By the recursive sieve method, we have revealed that the set T_e of all exponents of Mersenne primes has the structure of a particular order topology

$$\lim T_i' = T_e.$$

So that we consider the conjecture in the particular order topological space, which is generated naturally by the recursively sifting process (2.8), rather than in real analysis, a quantitative model of prime sets or heuristics model.

We consider all sequences of finite sets (G_i), such that $A_i' \subset G_i \subset L_i$, they converge to the end sifted set T_e from set theory

$$\lim G_i = T_e.$$

We try to endow all set-theoretical convergences $\lim G_i = T_e$ with an order topology using the recursively sifting process (2.8) as an order relation, then construct a particular order topological space $\mathbf{G}$.

Here we must be careful about the existence of order topological limits.

First, according to the definition, there exists no order topology on the empty set or sets with a single element[7] [8].

Next, we quote topologist L.D. Kudryavtsev's the existence of limits of a function for $\lim |G_i|$.

If the space $\mathbf{X}$ satisfies the first axiom of countability at the point T_e and the space $\mathbf{Y}$ is Hausdorff, then for existence of the limit $\lim |G_i|$ of the cardinal function $|G_i|$, it is necessary and sufficient that for any sequence (G_i) , such that $\lim G_i = T_e$, the limit $\lim |G_i|$ exists. If this condition holds, the limit $\lim |G_i|$ does not depend on the choice of the sequence (G_i) , and the common value of these limits is the limit of $(|G_i|)$ at T [6].

Note, we consider the set sequences; the empty $\emptyset$ is as an element. If we find out at least one prime pattern, then the set sequence (A_i') has more one element.

Only if the end sifted set is empty $T_e = \emptyset$, the limits $\lim A_i'$ and $\lim |A_i'|$ have no existence. The existence of all other limits $\lim G_i$, $\lim |G_i|$ is not sufficient from the above general topology. Thus at the point $T_e = \emptyset$, there is no "continuous" or "non-continuous". There is no contradiction. In this case, one needs no order topology.

If the end sifted set is not empty $T_e \neq \emptyset$, since the inclusion relation $G_i \supset A_i'$, the sequence (A_i') has more one element, every sequence (G_i) has more one element, every limit of set theory $\lim G_i$ may be endowed with an order topology. Every limit of order topology $\lim G_i$ has existence, every limit of order topology $\lim |G_i|$ has existence. The condition of existence of $\lim |T_i'|$, $\lim T_i'$, $\lim |A_i'|$, $\lim A_i'$ is sufficient. Thus our proof of the theorem 3.1 and 3.2 is correct.

In the formal system $P(N)$ we deal with the Ross-Littwood paradox and find out a proof of the Mersenne prime conjecture in the particular order topological space.

The Ross-Littwood paradox shows that the restricted definition for order topology, assume X has more one element, is necessary.

We consider the set of various prime patterns T_e, in advance, we have known at least one prime pattern, and in advance, we have known that the end sifted set is not empty $T_e \neq \emptyset$.

By the same paradigm, we may predicate whether another prime pattern will persist for ever or not.

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