The Paradigm of Complex Probability and the Central Limit Theorem

Abdo Abou Jaoudé

Notre Dame University - Louaize

Abstract- The concept of mathematical probability was established in 1933 by Andrey Nikolaevich Kolmogorov by defining a system of five axioms. This system can be enhanced to encompass the imaginary numbers set after the addition of three novel axioms. As a result, any random experiment can be executed in the complex probabilities set $\mathcal{C}$ which is the sum of the real probabilities set $\mathcal{R}$ and the imaginary probabilities set $\mathcal{M}$. We aim here to incorporate supplementary imaginary dimensions to the random experiment occurring in the "real" laboratory in $\mathcal{R}$ and therefore to compute all the probabilities in the sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$. Accordingly, the probability in the whole set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is constantly equivalent to one independently of the distribution of the input random variable in $\mathcal{R}$, and subsequently the output of the stochastic experiment in $\mathcal{R}$ can be determined absolutely in $\mathcal{C}$. This is the consequence of the fact that the probability in $\mathcal{C}$ is computed after the subtraction of the chaotic factor from the degree of our knowledge of the nondeterministic experiment. We will apply this innovative paradigm to the well-known Central Limit Theorem and to prove as well its convergence in a novel way.

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Author: Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Notre Dame University-Louaize, Lebanon. Email: abdoaj@idm.net.lb

NOMENCLATURE

$\mathcal{R}$ = real set of events
$\mathcal{M}$ = imaginary set of events
$\mathcal{C}$ = complex set of events
i = the imaginary number where $i = \sqrt{-1}$ or $i^2 = -1$
EKA = Extended Kolmogorov's Axioms
CPP = Complex Probability Paradigm
P_{rob} = probability of any event
P_r = probability in the real set $\mathcal{R}$ = probability of convergence in $\mathcal{R}$
P_m = probability in the imaginary set $\mathcal{M}$ corresponding to the real probability in $\mathcal{R}$ = probability of divergence in $\mathcal{M}$
Pc = probability of an event in $\mathcal{R}$ with its associated complementary event in $\mathcal{M}$ = probability in the complex probability set $\mathcal{C}$
z = complex probability number = sum of P_r and P_m = complex random vector
DOK = $|z|^2$ = the degree of our knowledge of the random system or experiment, it is the square of the norm of z
Chf = the chaotic factor of z
MChf = magnitude of the chaotic factor of z
n = number of random vectors = the random sample size

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S_n = the random sample mean of size n

Z = the resultant complex random vector = $\sum_{j=1}^{n} z_j$

$DOK_Z = \frac{|Z|^2}{n^2}$ = the degree of our knowledge of the whole stochastic system

$Chf_Z = \frac{Chf}{n^2}$ = the chaotic factor of the whole stochastic system

$MChf_Z$ = magnitude of the chaotic factor of the whole stochastic system

Z_U = the resultant complex random vector corresponding to a uniform random distribution

DOK_{Z_U} = the degree of our knowledge of the whole stochastic system corresponding to a uniform random distribution

Chf_{Z_U} = the chaotic factor of the whole stochastic system corresponding to a uniform random distribution

$MChf_{Z_U}$ = the magnitude of the chaotic factor of the whole stochastic system corresponding to a uniform random distribution

Pc_U = probability in the complex probability set $\mathcal{C}$ of the whole stochastic system corresponding to a uniform random distribution

CLT = Central Limit Theorem

I. Introduction

Firstly, in this introductory section an overview of the central limit theorem will be done. In probability theory, the central limit theorem (CLT) establishes that, in some situations, when independent random variables are added, their properly normalized sum tends toward a normal distribution (informally a bell-shaped curve) even if the original variables themselves are not normally distributed. The theorem is a key concept in probability theory because it implies that probabilistic and statistical methods that work for normal distributions can be applicable to many problems involving other types of distributions.

Mathematically, if $x_1, x_2, \ldots, x_n$ is a random sample of size n taken from a population with mean μ and finite variance σ^2 and if S_n is the sample mean, the limiting form of the distribution of $\Xi = \left(\frac{S_n - \mu}{\sigma / \sqrt{n}}\right)$ as $n \to +\infty$, is the standard normal distribution $N(0,1) = \Phi(\xi / \sigma)$ [1]. For example, suppose that a sample is obtained containing many observations, each observation being randomly generated in a way that does not depend on the values of the other observations, and that the arithmetic mean of the observed values is computed. If this procedure is performed many times, the central limit theorem says that the probability distribution of the average will closely approximate a normal distribution. A simple example of this is that if one flips a coin many times, the probability of getting a given number of heads will approach a normal distribution, with the mean equal to half the total number of flips. At the limit of an infinite number of flips, it will equal a normal distribution.

The central limit theorem has several variants. In its common form, the random variables must be identically distributed. In variants, convergence of the mean to the normal distribution also occurs for non-identical distributions or for non-independent observations, if they comply with certain conditions. The earliest version of this theorem, that the normal distribution may be used as an approximation to the binomial distribution, is the De Moivre–Laplace theorem.

The Dutch mathematician Henk Tijms writes [2]:
“The central limit theorem has an interesting history. The first version of this theorem was postulated by the French-born mathematician Abraham De Moivre who, in a remarkable article published in 1733, used the normal distribution to approximate the distribution of the number of heads resulting from many tosses of a fair coin. This finding was far ahead of its time, and was nearly forgotten until the famous French mathematician Pierre-Simon Laplace rescued it from obscurity in his monumental work “Théorie analytique des probabilités”, which was published in 1812. Laplace expanded De Moivre's finding by approximating the binomial distribution with the normal distribution. But as with De Moivre, Laplace's finding received little attention in his own time. It was not until the nineteenth century was at an end that the importance of the central limit theorem was discerned, when, in 1901, Russian mathematician Aleksandr Lyapunov defined it in general terms and proved precisely how it worked mathematically. Nowadays, the central limit theorem is considered to be the unofficial sovereign of probability theory.”

Moreover, Sir Francis Galton described the Central Limit Theorem in this way [3]:
“I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error". The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.”

Additionally, the actual term "Central Limit Theorem" (in German: "zentraler Grenzwertsatz") was first used by George Pólya in 1920 in the title of a paper [4,5]. Pólya referred to the theorem as "central" due to its importance in probability theory. According to Le Cam, the French school of probability interprets the word central in the sense that "it describes the behavior of the center of the distribution as opposed to its tails" [5]. The abstract of the paper On the central limit theorem of calculus of probability and the problem of moments by Pólya [4] in 1920 translates as follows:
“The occurrence of the Gaussian probability density $1 = e^{-x^2}$ in repeated experiments, in errors of measurements, which result in the combination of very many and very small elementary errors, in diffusion processes etc., can be explained, as is well-known, by the very same limit theorem, which plays a central role in the calculus of probability. The actual discoverer of this limit theorem is to be named Laplace; it is likely that its rigorous proof was first given by Tschebyscheff and its sharpest formulation can be found, as far as I am aware of, in an article by Liapounoff. ...”

A thorough account of the theorem's history, detailing Laplace's foundational work, as well as Cauchy's, Bessel's and Poisson's contributions, is provided by Hald [6]. Two historical accounts, one covering the development from Laplace to Cauchy, the second the contributions by von Mises, Pólya, Lindeberg, Lévy, and Cramér during the 1920s, are given by Hans Fischer [7]. Le Cam describes a period around 1935 [5]. Bernstein [8] presents a historical discussion focusing on the work of Pafnuty Chebyshev and his students Andrey Markov and Aleksandr Lyapunov that led to the first proofs of the *CLT* in a general setting.

Through the 1930s, progressively more general proofs of the Central Limit Theorem were presented. Many natural systems were found to exhibit Gaussian distributions—a typical example being height distributions for humans. When statistical methods such as analysis of variance became established in the early 1900s, it became increasingly common to assume underlying Gaussian distributions [9].

A curious footnote to the history of the Central Limit Theorem is that a proof of a result similar to the 1922 Lindeberg *CLT* was the subject of Alan Turing's 1934 Fellowship Dissertation for King's

College at the University of Cambridge. Only after submitting the work did Turing learn it had already been proved. Consequently, Turing's dissertation was not published [10-22].

Finally, and to conclude, this research work is organized as follows: After the introduction in section I, the purpose and the advantages of the present work are presented in section II. Afterward, in section III, we will summarize the complex probability paradigm with its original parameters and with a brief interpretation. In section IV, the De Moivre–Laplace theorem will be explained. In section V, the Poisson theorem will be clarified. In section VI, the Central Limit Theorem will be presented. In section VII, I will extend the Central Limit Theorem to the imaginary and complex probability sets and hence link this concept to my novel complex probability paradigm. Moreover, in this section, I will prove the convergence in *CLT* using the concept of the resultant complex random vector *Z*. Furthermore, in section VIII a flowchart of the complex probability paradigm and *CLT* prognostic model will be drawn. Additionally, in section IX, the simulations of *CLT* will be accomplished in the discrete and continuous cases. Finally, I conclude the work by doing a comprehensive summary in section X, and then present the list of references cited in the current research work.

II. The Purpose and the Advantages of The Present Work

In this section we will present the purpose and the advantages of the current research work. Calculating probabilities is the crucial task of classical probability theory. Adding supplementary dimensions to nondeterministic experiments will yield a deterministic expression of the theory of probability. This is the novel and original idea at the foundations of my complex probability paradigm. As a matter of fact, probability theory is a stochastic system of axioms in its essence; that means that the phenomena outputs are due to randomness and chance. By adding novel imaginary dimensions to the nondeterministic phenomenon happening in the set $\mathcal{R}$ will lead to a deterministic phenomenon and thus a stochastic experiment will have a certain output in the complex probability set $\mathcal{C}$. If the chaotic experiment becomes completely predictable then we will be fully capable to predict the output of random events that arise in the real world in all stochastic processes. Accordingly, the task that has been achieved here was to extend the random real probabilities set $\mathcal{R}$ to the deterministic complex probabilities set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and this by incorporating the contributions of the set $\mathcal{M}$ which is the complementary imaginary set of probabilities to the set $\mathcal{R}$. Consequently, since this extension reveals to be successful, then an innovative paradigm of stochastic sciences and prognostic was put forward in which all nondeterministic phenomena in $\mathcal{R}$ was expressed deterministically in $\mathcal{C}$. I coined this novel model by the term "The Complex Probability Paradigm" that was initiated and established in my fourteen earlier research works. [23-36]

Accordingly, the advantages and the purpose of the present paper are to:

1- Extend the theory of classical probability to cover the complex numbers set, hence to connect the probability theory to the field of complex variables and analysis. This task was started and elaborated in my earlier fourteen papers.

2- Apply the novel probability axioms and paradigm to the *CLT*.

3- Show that all nondeterministic phenomena can be expressed deterministically in the complex probabilities set which is $\mathcal{C}$.

4- Compute and quantify both the degree of our knowledge and the chaotic factor in *CLT*.

5- Represent and show the graphs of the functions and parameters of the innovative paradigm related to *CLT*.

6- Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities; hence, no chaos, no randomness, no ignorance, no uncertainty, no unpredictability, no nondeterminism, and no disorder exist in:

$$\mathcal{C}\text{ (complex set)} = \mathcal{R}\text{ (real set)} + \mathcal{M}\text{ (imaginary set)}.$$

7- Prove the convergence in the stochastic *CLT* in an original way by using the newly defined axioms and paradigm.

8- Pave the way to implement this inventive model to other topics in prognostics and to the field of stochastic processes. These will be the goals of my future research works.

Concerning some applications of the original elaborated paradigm and as a future work, it can be applied to any random phenomena using *CLT* methods whether in the discrete or in the continuous cases.

Furthermore, compared with existing literature, the main contribution of the present research work is to apply the novel paradigm of complex probability to the concepts and techniques of the stochastic *CLT* methods and simulations as well as to prove the convergence in *CLT* in a novel and original way. The next figure illustrates the major purposes and objectives of the Complex Probability Paradigm (*CPP*) (Figure 1).

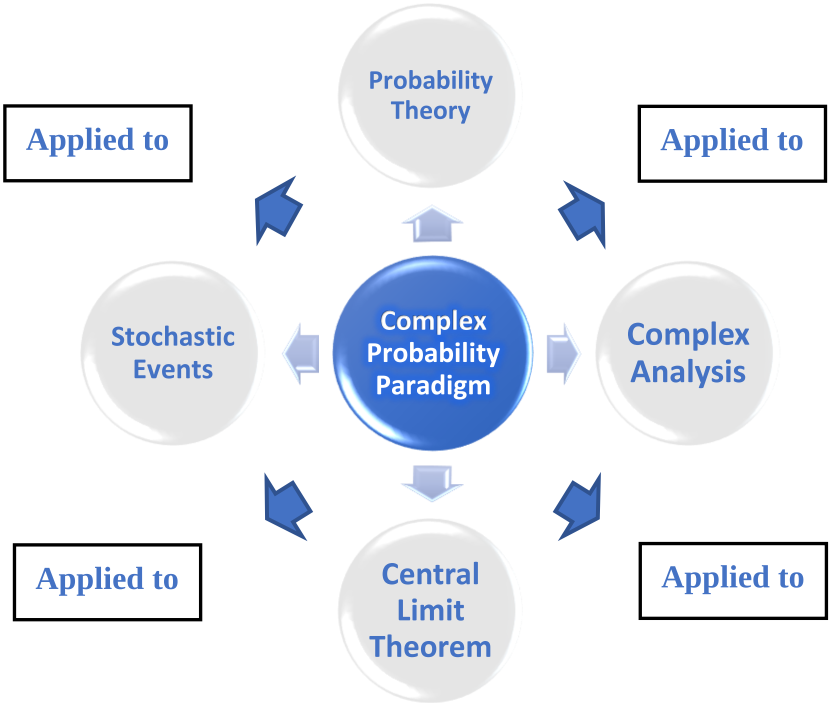


Figure 1: The diagram of the main purposes of the Complex Probability Paradigm

III. The Complex Probability Paradigm [23-36] [37-68]

3.1 The Original Andrey Nikolaevich Kolmogorov System of Axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events. The five axioms for a finite set E are:

Axiom 1: *F* is a field of sets.
Axiom 2: *F* contains the set *E*.
Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of *A*, is assigned to each set *A* in *F*. We have always $0 \le P_{rob}(A) \le 1$.
Axiom 4: $P_{rob}(E)$ equals 1.
Axiom 5: If *A* and *B* have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that *A* and *B* are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both *A* and *B* are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for *N* disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}\left(A_j\right)$$

And we say also that for *N* independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}\left(A_j\right)$$

3.2 Adding the Imaginary Part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part) to the event *A* in $\mathcal{R}$ (the real part). It follows that $P_r + P_m / i = 1$ where *i* is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.
Axiom 7: We construct the complex number or vector $Z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|Z|$ such that:

$$|Z|^2 = P_r^2 + (P_m / i)^2 .$$

Axiom 8: Let *Pc* denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that *Pc* is the probability of an event *A* in $\mathcal{R}$ with its associated complementary event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m \text{ and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers.

3.3 A Brief Interpretation of the Novel Paradigm

To summarize the novel paradigm, we state that in the real probability universe $\mathcal{R}$ our degree of our certain knowledge is undesirably imperfect and hence unsatisfactory, thus we extend our analysis to the set of complex numbers $\mathcal{C}$ which incorporates the contributions of both the set of real probabilities which is $\mathcal{R}$ and the complementary set of imaginary probabilities which is $\mathcal{M}$. Afterward, this will yield an absolute and perfect degree of our knowledge in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ because $Pc = 1$ constantly. As a matter of fact, the work in the complex universe $\mathcal{C}$ gives way to a sure prediction of any stochastic experiment, because in $\mathcal{C}$ we remove and subtract from the computed degree of our knowledge the measured chaotic factor. This will generate a probability in the universe $\mathcal{C}$ equal to 1 ($Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$). Many illustrations taking into consideration numerous continuous and discrete probability distributions in my fourteen previous research papers confirm this hypothesis and innovative paradigm [23-36]. The Extended Kolmogorov Axioms (*EKA* for short) or the Complex Probability Paradigm (*CPP* for short) can be shown and summarized in the next illustration (Figure 2):

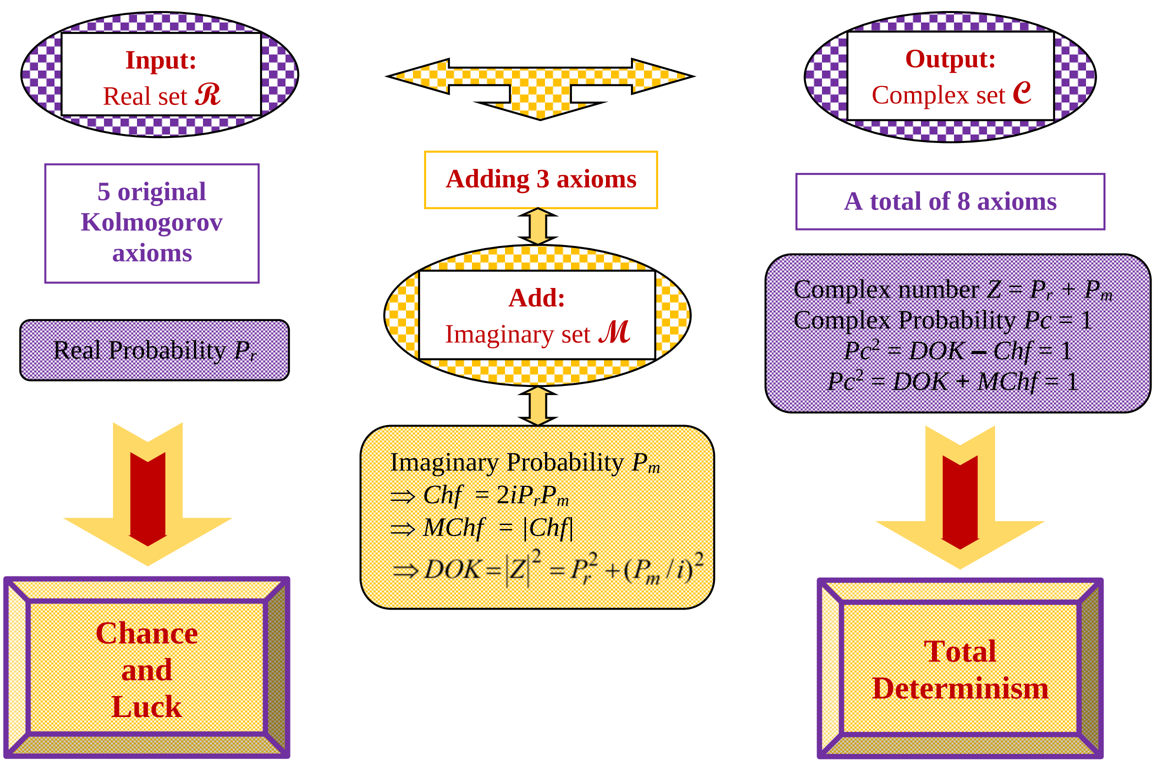


Figure 2: The *EKA* or the *CPP* summarized illustration

IV. The De Moivre–Laplace Theorem [69-71]

In probability theory, the De Moivre–Laplace theorem, which is a special case of the central limit theorem, states that the normal distribution may be used as an approximation to the binomial distribution under certain conditions. In particular, the theorem shows that the probability mass function of the random number of "successes" observed in a series of n independent Bernoulli trials, each having a probability p of success and a probability $q = 1 - p$ of failure (a binomial distribution with n trials), converges to the probability density function of the normal distribution with mean np and standard deviation $\sqrt{np(1-p)} = \sqrt{npq}$, as n grows large, assuming p is not 0 or 1.

The theorem appeared in the second edition of The Doctrine of Chances by Abraham De Moivre, published in 1738. Although De Moivre did not use the term "Bernoulli trials", he wrote about the probability distribution of the number of times "heads" appears when a coin is tossed 3600 times.

This is one derivation of the particular Gaussian function used in the normal distribution.

Mathematically, as n grows large, for k in the neighborhood of np we can approximate

$$P_{rob}(X=k)=\binom{n}{k}p^k q^{n-k}\simeq\frac{1}{\sqrt{2\pi npq}}e^{-\frac{(k-np)^2}{2npq}}$$

Where $p+q=1, \quad p,q>0$

And $\binom{n}{k}={}_nC_k=C(n,k)=\frac{n!}{k!(n-k)!}$ is the binomial coefficient.

In the sense that the ratio of the left-hand side to the right-hand side converges to 1 as $n\to+\infty$.

V. The Poisson Distribution and the *CLT* [72-75]

In probability theory and statistics, the Poisson distribution, named after the French mathematician Siméon Denis Poisson is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event. The Poisson distribution can also be used for the number of events in other specified intervals such as distance, area or volume. The Poisson distribution is popular for modeling the number of times an event occurs in an interval of time or space.

A discrete random variable X is said to have a Poisson distribution with parameter $\lambda>0$, if, for $k=0,1,2,\ldots,+\infty$, the probability mass function of X is given by:

$$f(k;\lambda)=P_{rob}(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$$

Where,
e is Euler's number or the basis of logarithms (*e* = 2.71828...)
and $k!$ is the factorial of *k*.

The positive real number λ is equal to the expected value of X and also to its variance:

$$\lambda=\mathrm{E}(X)=\mathrm{Var}(X)$$

The Poisson distribution can be applied to systems with a large number of possible events, each of which is rare. The number of such events that occur during a fixed time interval is, under the right circumstances, a random number with a Poisson distribution.

The Poisson distribution can be derived as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed. Therefore, it can be used as an approximation of the binomial distribution if *n* is sufficiently large and *p* is sufficiently small. There is a rule of thumb stating that the Poisson distribution is a good approximation of the binomial distribution if n is at least 20 and *p* is smaller than or equal to 0.05, and an excellent approximation if $n\geq100$ and $np\leq10$. The cumulative distribution functions of the Poisson and Binomial distributions are related in the following way:

$$F_{\text{Binomial}}(k;n,p) \simeq F_{\text{Poisson}}(k;\lambda = np)$$

For sufficiently large values of λ, (say $\lambda > 1000$), the normal distribution with mean λ and variance λ (standard deviation = $\sqrt{\lambda}$) is an excellent approximation to the Poisson distribution. If λ is greater than about 10, then the normal distribution is a good approximation if an appropriate continuity correction is performed. The cumulative distribution functions of the Poisson and Normal distributions are related in the following way:

$$F_{\text{Poisson}}(x;\lambda = np) \simeq F_{\text{Normal}}(x;\mu = \lambda, \sigma^2 = \lambda)$$

VI. The Classical Central Limit Theorem [13]

Let $\{x_1, x_2, \ldots, x_n\}$ be a random sample of size n, that is, a sequence of independent and identically distributed random variables drawn from a distribution of expected value given by μ and finite variance given by σ^2. Suppose we are interested in the sample average which is:

$$S_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

of these random variables. By the law of large numbers, the sample averages converge in probability and almost surely to the expected value μ as $n \to +\infty$. The classical central limit theorem describes the size and the distributional form of the stochastic fluctuations around the deterministic number μ during this convergence. More precisely, it states that as n gets larger, the distribution of the difference between the sample average S_n and its limit μ, when multiplied by the factor $\sqrt{n}$ (that is $\sqrt{n}(S_n - \mu)$), approximates the normal distribution with mean 0 and variance σ^2. For large enough n, the distribution of S_n is close to the normal distribution with mean μ and variance σ^2 / n hence with a standard deviation $\sigma / \sqrt{n}$. The usefulness of the theorem is that the distribution of $\sqrt{n}(S_n - \mu)$ approaches normality regardless of the shape of the distribution of the individual x_j. Formally, the theorem can be stated as follows:

Lindeberg–Lévy *CLT*: Suppose $\{x_1, x_2, \ldots, x_n\}$ is a sequence of independent and identically distributed random variables with $\mathrm{E}[x_j] = \mu$ and $\mathrm{Var}[x_j] = \sigma^2 < +\infty$, $\forall j: \ 1 \le j \le n$.

Then as n approaches infinity, the random variables $\sqrt{n}(S_n - \mu)$ converge in distribution to a normal distribution $N(0, \sigma^2)$:

$$\sqrt{n}(S_n - \mu) \to N(0, \sigma^2).$$

In the case $\sigma > 0$, convergence in distribution means that the cumulative distribution functions of $\sqrt{n}(S_n - \mu)$ converge pointwise to the cumulative distribution function (*CDF*) of the $N(0, \sigma^2)$ distribution: for every real number ξ,

$$\lim_{n \to +\infty} P_{rob}\left[\sqrt{n}(S_n - \mu) \le \xi\right] = \lim_{n \to +\infty} P_{rob}\left[\frac{\sqrt{n}(S_n - \mu)}{\sigma} \le \frac{\xi}{\sigma}\right] = \Phi\left(\frac{\xi}{\sigma}\right)$$

where $\Phi(\xi)$ is the standard normal *CDF* evaluated at ξ. The convergence is uniform in ξ in the sense that:

$$\lim_{n\to+\infty} \sup_{\xi\in R} \left| P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right] - \Phi\left(\frac{\xi}{\sigma}\right) \right| = 0$$

where 'sup' denotes the least upper bound (or supremum) of the set.

VII. The Resultant Complex Random Vector *Z* of *CPP* and the Central Limit Theorem [23-36] [76-95]

A powerful tool will be described in the current section which was developed in my personal previous research papers and which is founded on the concept of a complex random vector that is a vector combining the real and the imaginary probabilities of a random outcome, defined in the three added axioms of *CPP* by the term $z_j = P_{rj} + P_{mj}$. Accordingly, we will define the vector *Z* as the resultant complex random vector which is the sum of all the complex random vectors z_j in the complex probability plane $\mathcal{C}$. This procedure is illustrated by considering first a general Bernoulli distribution, then we will discuss a discrete probability distribution with *n* equiprobable random vectors as a general case. In fact, if *z* represents one vector from the uniform distribution *U*, then Z_U represents the whole system of vectors from the uniform distribution *U* that means the whole random distribution in the complex probability plane $\mathcal{C}$. So, it follows directly that a Bernoulli distribution can be understood as a simplified system or population with two random variables (section 7.1), whereas the general case is a random system or population with *n* random variables (section 7.2). Afterward, I will prove the convergence in *CLT* using this new powerful concept and tool (section 7.3).

7.1 The Resultant Complex Random Vector Z of a General Bernoulli Distribution (A Distribution with Two Random Variables)

First, let us consider the following general Bernoulli distribution and let us define its complex random vectors and their resultant (Table 1):

Table 1: A general Bernoulli distribution in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

Outcome	x_j	x_1	x_2
In $\mathcal{R}$	P_{rj}	$P_{r1} = p$	$P_{r2} = q$
In $\mathcal{M}$	P_{mj}	$P_{m1} = i(1-p) = iq$	$P_{m2} = i(1-q) = ip$
In $\mathcal{C} = \mathcal{R} + \mathcal{M}$	z_j	$z_1 = P_{r1} + P_{m1}$	$z_2 = P_{r2} + P_{m2}$

Where,

x_1 and x_2 are the outcomes of the first and second random vectors respectively.

P_{r1} and P_{r2} are the real probabilities of x_1 and x_2 respectively.

P_{m1} and P_{m2} are the imaginary probabilities of x_1 and x_2 respectively.

We have

$$\sum_{j=1}^{2} P_{rj} = P_{r1} + P_{r2} = p + q = 1$$

and

$$\sum_{j=1}^{2} P_{mj} = P_{m1} + P_{m2} = iq + ip = i(1-p) + ip$$
$$= i - ip + ip = i = i(2-1) = i(n-1)$$

Where n is the number of random vectors or outcomes which is equal to 2 for a Bernoulli distribution.

The complex random vector corresponding to the random outcome x_1 is:

$$z_1 = P_{r1} + P_{m1} = p + i(1-p) = p + iq$$

The complex random vector corresponding to the random outcome x_2 is:

$$z_2 = P_{r2} + P_{m2} = q + i(1-q) = q + ip$$

The resultant complex random vector is defined as follows:

$$Z = \sum_{j=1}^{2} z_j = z_1 + z_2 = \sum_{j=1}^{2} P_{rj} + \sum_{j=1}^{2} P_{mj}$$
$$= (p+iq) + (q+ip) = (p+q) + i(p+q)$$
$$= 1 + i = 1 + i(2-1)$$
$$\Rightarrow Z = 1 + i(n-1)$$

The probability Pc_1 in the complex plane $\mathcal{C} = \mathcal{R} + \mathcal{M}$ which corresponds to the complex random vector z_1 is computed as follows:

$$|z_1|^2 = P_{r1}^2 + (P_{m1}/i)^2 = p^2 + q^2$$
$$Chf_1 = -2P_{r1}P_{m1}/i = -2pq$$
$$\Rightarrow Pc_1^2 = |z_1|^2 - Chf_1$$
$$= p^2 + q^2 + 2pq = (p+q)^2 = 1^2 = 1$$
$$\Rightarrow Pc_1 = 1$$

This is coherent with the three novel complementary axioms defined for the *CPP*.

Similarly, Pc_2 corresponding to z_2 is:

$$|z_2|^2 = P_{r2}^2 + (P_{m2}/i)^2 = q^2 + p^2$$
$$Chf_2 = -2P_{r2}P_{m2}/i = -2qp$$
$$\Rightarrow Pc_2^2 = |z_2|^2 - Chf_2$$
$$= q^2 + p^2 + 2qp = (q+p)^2 = 1^2 = 1$$
$$\Rightarrow Pc_2 = 1$$

The probability Pc in the complex plane $\mathcal{C}$ which corresponds to the resultant complex random vector $Z = 1 + i$ is computed as follows:

$$|Z|^2 = \left(\sum_{j=1}^{2} P_{rj}\right)^2 + \left(\sum_{j=1}^{2} P_{mj} / i\right)^2 = 1^2 + 1^2 = 2$$

$$Chf = -2\sum_{j=1}^{2} P_{rj} \sum_{j=1}^{2} P_{mj} / i = -2(1)(1) = -2$$

$$\text{Let } s^2 = |Z|^2 - Chf = 2 + 2 = 4 \Rightarrow s = 2$$

$$\Rightarrow Pc^2 = \frac{s^2}{n^2} = \frac{|Z|^2 - Chf}{n^2} = \frac{|Z|^2}{n^2} - \frac{Chf}{n^2} = \frac{4}{2^2} = \frac{4}{4} = 1$$

$$\Rightarrow Pc = \frac{s}{n} = \frac{2}{2} = 1$$

Where s is an intermediary quantity used in our computation of Pc.

Pc is the probability corresponding to the resultant complex random vector Z in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and is also equal to 1. Actually, Z represents both z_1 and z_2 that means the whole distribution of random vectors of the general Bernoulli distribution in the complex plane $\mathcal{C}$ and its probability Pc is computed in the same way as Pc_1 and Pc_2.

By analogy, for the case of one random vector z_j we have:

$$Pc_j^2 = |z_j|^2 - Chf_j \quad \text{with} \ (n = 1).$$

In general, for the vector Z we have:

$$Pc^2 = \frac{|Z|^2}{n^2} - \frac{Chf}{n^2}; \quad (n \geq 1)$$

Where the degree of our knowledge of the whole distribution is equal to $DOK_Z = \frac{|Z|^2}{n^2}$, its relative chaotic factor is $Chf_Z = \frac{Chf}{n^2}$, and its relative magnitude of the chaotic factor is $MChf_Z = |Chf_Z|$

Notice, if $n = 1$ in the previous formula, then:

$$Pc^2 = \frac{|Z|^2}{n^2} - \frac{Chf}{n^2} = DOK_Z - Chf_Z = \frac{|Z|^2}{1^2} - \frac{Chf}{1^2} = |Z|^2 - Chf = |z_j|^2 - Chf_j = Pc_j^2$$

which is coherent with the calculations already done.

To illustrate the concept of the resultant complex random vector Z, I will use the following graph (Figure 3).

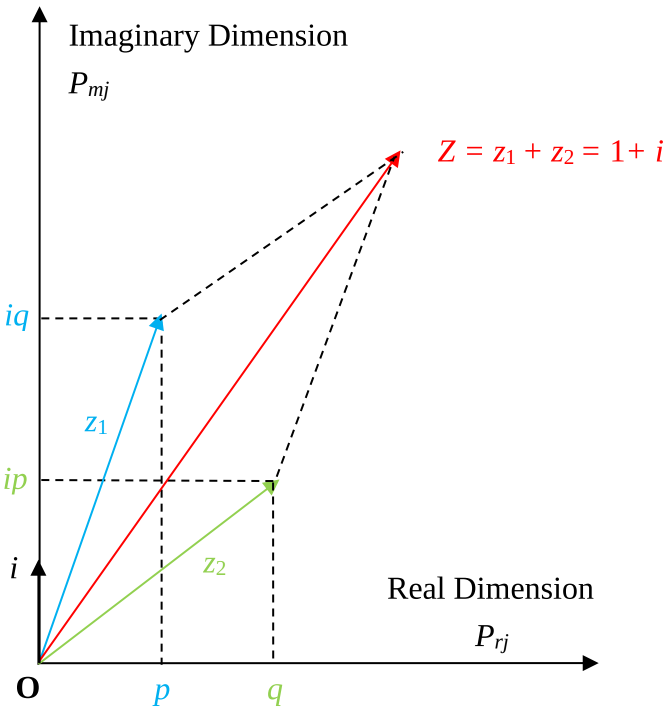


Figure 3: The resultant complex random vector $Z = z_1 + z_2$ for a general Bernoulli distribution in the complex probability plane $\mathcal{C}$

7.2 The General Case: A Discrete Distribution with n Equiprobable Random Vectors (A Uniform Distribution U with n Random Variables)

As a general case, let us consider then this discrete probability distribution with n equiprobable random vectors which is a discrete uniform probability distribution U. In fact, let $\{x_1, x_2, \ldots, x_n\}$ be a random sample of size n, that is, a sequence of independent and identically distributed random variables drawn from a distribution or a population of expected value given by μ and finite variance given by σ^2. Since all random variables have an equal probability to be chosen in the sample from the population then we have a discrete uniform probability distribution U (Table 2):

Table 2: A discrete uniform distribution with n equiprobable random vectors in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

Outcome	x_j	x_1	x_2	$\ldots$	x_n
In $\mathcal{R}$	P_{rj}	$P_{r1} = \frac{1}{n}$	$P_{r2} = \frac{1}{n}$	$\ldots$	$P_{rn} = \frac{1}{n}$
In $\mathcal{M}$	P_{mj}	$P_{m1} = i\left(1 - \frac{1}{n}\right)$	$P_{m2} = i\left(1 - \frac{1}{n}\right)$	$\ldots$	$P_{mn} = i\left(1 - \frac{1}{n}\right)$
In $\mathcal{C} = \mathcal{R} + \mathcal{M}$	z_j	$z_1 = P_{r1} + P_{m1}$	$z_2 = P_{r2} + P_{m2}$	$\ldots$	$z_n = P_{rn} + P_{mn}$

We have here in $\mathcal{C} = \mathcal{R} + \mathcal{M}$:

$$z_j = P_{rj} + P_{mj}, \quad \forall j: \; 1 \le j \le n,$$

$$\text{and } \; z_1 = z_2 = \ldots = z_n = \frac{1}{n} + \frac{i(n-1)}{n}$$

$$\Rightarrow Z_U = \sum_{j=1}^{n} z_j = z_1 + z_2 + \ldots + z_n = nz_j = n\left(\frac{1}{n} + \frac{i(n-1)}{n}\right) = 1 + i(n-1)$$

Moreover, we can notice that: $|z_1| = |z_2| = \cdots = |z_n|$, hence,

$$|Z_U| = |z_1 + z_2 + \ldots + z_n| = n|z_1| = n|z_2| = \ldots = n|z_n|$$

$$\Rightarrow |Z_U|^2 = n^2|z_j|^2 = n^2\left(\frac{1}{n^2} + \frac{(n-1)^2}{n^2}\right) = 1 + (n-1)^2, \quad \text{where } 1 \le j \le n;$$

And

$$Chf = n^2 \times Chf_j = -2 \times P_{rj} \times (P_{mj}/i) \times n^2 = -2n^2 \times \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right) = -2(1)(n-1) = -2(n-1)$$

$$\Rightarrow s^2 = |Z_U|^2 - Chf = 1 + (n-1)^2 + 2(n-1) = [1 + (n-1)]^2 = n^2$$

$$\Rightarrow Pc_U{}^2 = \frac{s^2}{n^2} = \frac{n^2}{n^2} = 1$$

$$= \frac{|Z_U|^2}{n^2} - \frac{Chf}{n^2} = \frac{1 + (n-1)^2}{n^2} - \frac{-2(n-1)}{n^2} = \frac{1 + (n-1)^2 + 2(n-1)}{n^2} = \frac{[1 + (n-1)]^2}{n^2} = \frac{n^2}{n^2} = 1$$

$$\Rightarrow Pc_U = 1$$

Where s is an intermediary quantity used in our computation of Pc_U.

Therefore, the degree of our knowledge corresponding to the resultant complex vector Z_U representing the whole uniform distribution is:

$$DOK_{Z_U} = \frac{|Z_U|^2}{n^2} = \frac{1 + (n-1)^2}{n^2},$$

and its relative chaotic factor is:

$$Chf_{Z_U} = \frac{Chf}{n^2} = -\frac{2(n-1)}{n^2},$$

Similarly, its relative magnitude of the chaotic factor is:

$$MChf_{Z_U} = |Chf_{Z_U}| = \left|\frac{Chf}{n^2}\right| = \left|-\frac{2(n-1)}{n^2}\right| = \frac{2(n-1)}{n^2}.$$

Thus, we can verify that we have always:

$$Pc_U^2 = \frac{|Z_U|^2}{n^2} - \frac{Chf}{n^2} = DOK_{Z_U} - Chf_{Z_U} = DOK_{Z_U} + MChf_{Z_U} = 1 \Leftrightarrow Pc_U = 1$$

What is important here is that we can notice the following fact. Take for example:

$$n=2 \Rightarrow DOK_{Z_U} = \frac{1+(2-1)^2}{2^2} = 0.5 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(2-1)}{2^2} = -0.5$$

$$n=4 \Rightarrow DOK_{Z_U} = \frac{1+(4-1)^2}{4^2} = 0.625 \geq 0.5 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(4-1)}{4^2} = -0.375 \geq -0.5$$

$$n=5 \Rightarrow DOK_{Z_U} = \frac{1+(5-1)^2}{5^2} = 0.68 \geq 0.625 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(5-1)}{5^2} = -0.32 \geq -0.375$$

$$n=10 \Rightarrow DOK_{Z_U} = \frac{1+(10-1)^2}{10^2} = 0.82 \geq 0.68 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(10-1)}{10^2} = -0.18 \geq -0.32$$

$$n=100 \Rightarrow DOK_{Z_U} = \frac{1+(100-1)^2}{100^2} = 0.9802 \geq 0.82 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(100-1)}{100^2} = -0.0198 \geq -0.18$$

$$n=1000 \quad \Rightarrow DOK_{Z_U} = \frac{1+(1000-1)^2}{1000^2} = 0.998002 \geq 0.9802 \quad \text{and}$$

$$Chf_{Z_U} = \frac{-2(1000-1)}{1000^2} = -0.001998 \geq -0.0198$$

$$n=1000000 \quad \Rightarrow DOK_{Z_U} = \frac{1+(1000000-1)^2}{(1000000)^2} = 0.999998 \geq 0.998002 \quad \text{and}$$

$$Chf_{Z_U} = \frac{-2(1000000-1)}{(1000000)^2} = -0.000001999998 \geq -0.001998$$

We can deduce mathematically using calculus that:

$$\lim_{n\to+\infty} \frac{|Z_U|^2}{n^2} = \lim_{n\to+\infty} DOK_{Z_U} = \lim_{n\to+\infty} \frac{1+(n-1)^2}{n^2} = 1,$$

$$\text{and} \lim_{n\to+\infty} \frac{Chf}{n^2} = \lim_{n\to+\infty} Chf_{Z_U} = \lim_{n\to+\infty} -\frac{2(n-1)}{n^2} = 0.$$

From the above, we can also deduce this conclusion:

As much as n increases, as much as the degree of our knowledge in $\mathcal{R}$ corresponding to the resultant complex vector is perfect and absolute, that means, it is equal to one, and as much as the chaotic factor that prevents us from foretelling exactly and totally the outcome of the stochastic phenomenon in $\mathcal{R}$ approaches zero. Mathematically we state that: If n tends to infinity then the degree of our knowledge in $\mathcal{R}$ tends to one and the chaotic factor tends to zero.

7.3 The Convergence in the CLT using Z and CPP

Let $\{x_1, x_2, \ldots, x_n\}$ be a random sample of size n, that is, a sequence of independent and identically distributed random variables drawn from a distribution of expected value given by μ and finite variance given by σ^2. Suppose we are interested in the sample average which is:

$$S_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

And let $P_r = P_{rob}(\text{Convergence in } CLT) = \dfrac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\dfrac{\xi}{\sigma}\right)}$

Subsequently, if $\lim\limits_{n\to+\infty} Chf_{Z_U} = 0$ then $\lim\limits_{n\to+\infty} Chf_{CLT} = 0$ (the Chaotic factor in *CLT*), therefore:

$$\Leftrightarrow \lim_{n\to+\infty} Chf_{CLT} = \lim_{n\to+\infty}\left[2iP_rP_m\right] = \lim_{n\to+\infty}\left[-2P_rP_m/i\right] = 0 \text{ since } i^2 = -1 \text{ hence } i = -\frac{1}{i}$$

$$\Leftrightarrow \begin{cases} P_r \to 0 \\ \text{OR} \\ P_m/i \to 0 \end{cases} \Leftrightarrow \begin{cases} P_r \to 0 \\ \text{OR} \\ P_r = 1 - P_m/i \to 1-0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} P_{rob}(\text{Convergence in } CLT) \to 0 \\ \text{OR} \\ P_{rob}(\text{Convergence in } CLT) \to 1 \end{cases}$$

that means:

1) either the simulation and the random sampling have not started yet that means:

$$P_r = P_{rob}(\text{Convergence in } CLT) = \frac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\dfrac{\xi}{\sigma}\right)} \to 0$$

$$\Leftrightarrow P_{rob}\left[\frac{\sqrt{n}\left(S_n-\mu\right)}{\sigma} \le \frac{\xi}{\sigma}\right] \to 0$$

2) or the *CLT* algorithm output and $\sqrt{n}\left(S_n-\mu\right)$ have converged that means:

$$P_r = P_{rob}(\text{Convergence in } CLT) = \frac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\dfrac{\xi}{\sigma}\right)} \to 1$$

$$\Leftrightarrow P_{rob}\left[\frac{\sqrt{n}\left(S_n-\mu\right)}{\sigma} \le \frac{\xi}{\sigma}\right] \to \Phi\left(\frac{\xi}{\sigma}\right)$$

that means also:

$$\lim_{n\to+\infty} \sup_{\xi\in R} \left| P_{rob}\left[\frac{\sqrt{n}\left(S_n-\mu\right)}{\sigma} \le \frac{\xi}{\sigma}\right] - \Phi\left(\frac{\xi}{\sigma}\right) \right| = 0$$

And $\sqrt{n}\left(S_n-\mu\right) \to N(0,\sigma^2)$, in other words, the random variables $\sqrt{n}\left(S_n-\mu\right)$ converge in distribution to a normal distribution $N(0,\sigma^2)$, as predicted by *CLT*.

This is due to the fact that $Chf_{CLT} = 0$ in only two places which are: $n = 0$ and $n \to +\infty$.

Additionally, if $\lim_{n\to+\infty} DOK_{Z_U} = 1$ then $\lim_{n\to+\infty} DOK_{CLT} = 1$ (the Degree of Our Knowledge in *CLT*), and since $Pc^2 = DOK - Chf = 1$ from *CPP*, therefore:

$$\Leftrightarrow \lim_{n\to+\infty} DOK_{CLT} = \lim_{n\to+\infty}\left[1+Chf_{CLT}\right] = 1 + \lim_{n\to+\infty} Chf_{CLT} = \lim_{n\to+\infty}\left[P_r^2 + \left(P_m / i\right)^2\right] = \lim_{n\to+\infty}\left[1 - 2P_rP_m / i\right] = 1$$

$$\Leftrightarrow \begin{cases} P_r \to 0 \\ \text{OR} \\ P_m / i \to 0 \end{cases} \Leftrightarrow \begin{cases} P_r \to 0 \\ \text{OR} \\ P_r = 1 - P_m / i \to 1 - 0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} P_{rob}(\text{Convergence in } CLT) \to 0 \\ \text{OR} \\ P_{rob}(\text{Convergence in } CLT) \to 1 \end{cases}$$

that means we have reached the same conclusions as above since $DOK_{CLT} = 1$ in only two places which are: $n = 0$ and $n \to +\infty$.

Furthermore, for $n = 1 \Rightarrow \frac{|Z|^2}{n^2} = DOK_{Z_U} = \frac{1+(1-1)^2}{1^2} = 1 \Rightarrow DOK_{CLT} = 1$

$$\text{And } \frac{Chf}{n^2} = Chf_{Z_U} = \frac{-2(1-1)}{1^2} = 0 \Rightarrow Chf_{CLT} = 0$$

This means that we have a random experiment or sample with only one outcome or vector, hence, either $P_r = 0$ (always diverging) or $P_r = 1$ (always converging), that means we have respectively either an impossible event or a sure event in $\mathcal{R}$. Consequently, we have surely the degree of our knowledge equal to one (perfect experiment knowledge) and the chaotic factor equal to zero (no chaos) since the random experiment is either respectively uncertain or certain which is absolutely logical.

Consequently, what we have done here is that we have proved the law of large numbers (already discussed in the published paper [28]) as well as the convergence in the *CLT* using *CPP*. In fact, as it is very well-known in the classical probability theory and statistics, the law of large numbers is tightly related and linked to the *CLT*. Here *CPP* comes and proves both of them in a novel and original way. The following figures (Figures 4 and 5) show the convergence of Chf_{Z_U} to 0 and of DOK_{Z_U} to 1 as functions of the size *n* of the random sample.

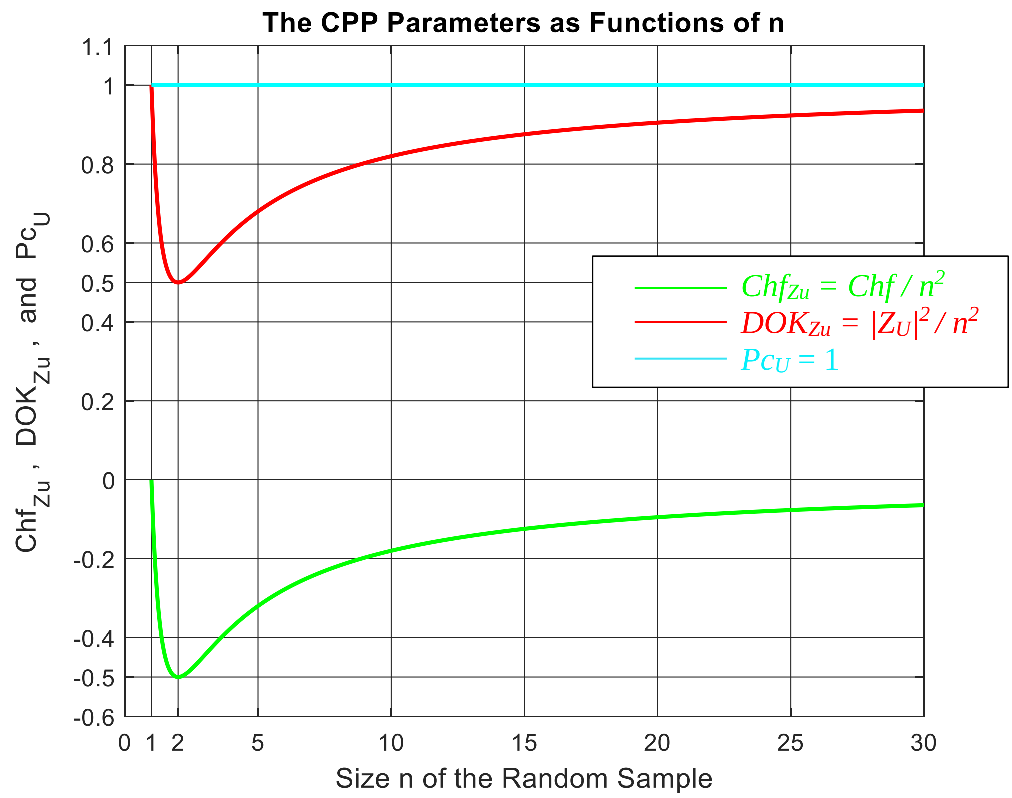


Figure 4: Chf_{Z_U} , DOK_{Z_U} , and Pc_U , as functions of n in 2D

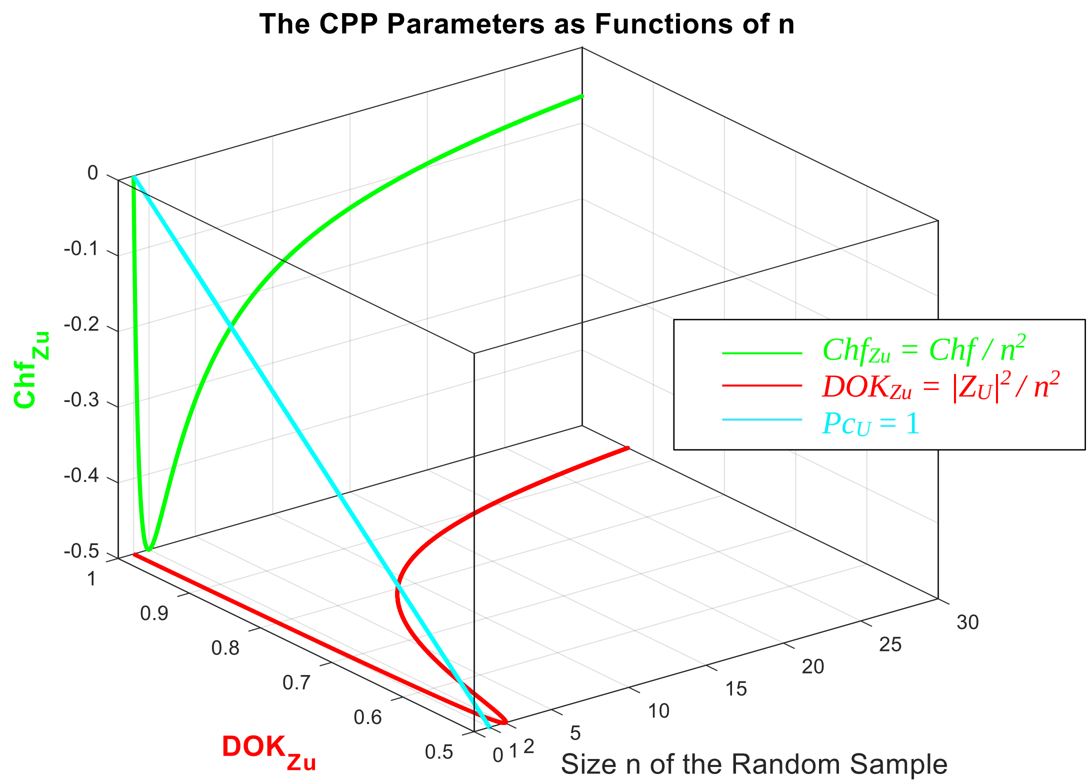
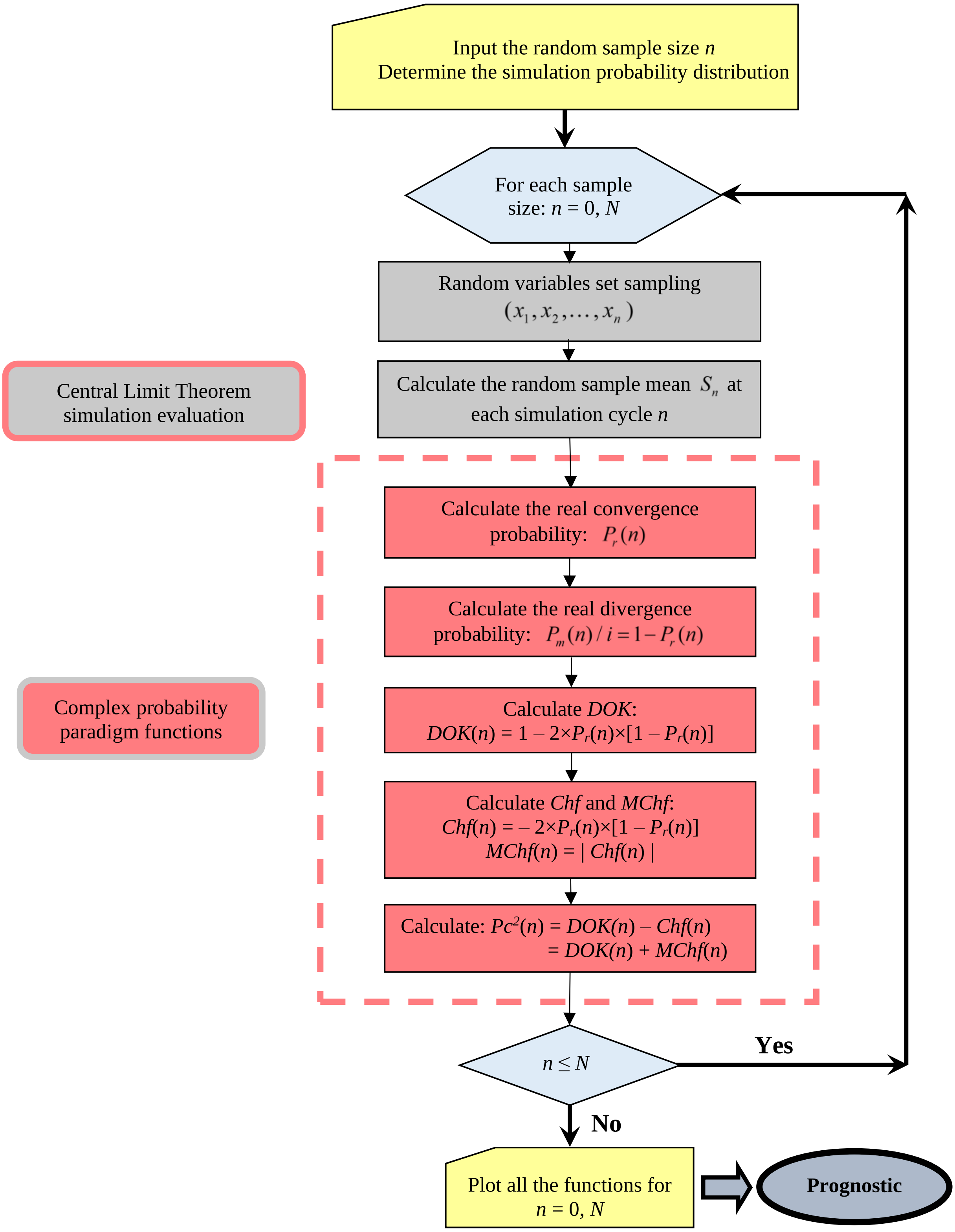


Figure 5: Chf_{Z_U} , DOK_{Z_U} , and Pc_U , as functions of n in 3D

VIII. Flowchart of the Complex Probability Paradigm and *CLT* Prognostic Model

The following flowchart summarizes all the procedures of the proposed complex probability paradigm prognostic model:



In fact, the proposed complex probability paradigm and prognostic model starts by determining the sample size and simulation cycles n taken from a population of observations. Then after determining the probability distribution taken into consideration (binomial, Poisson, Gaussian, Standard Normal, etc.) we apply accordingly the central limit theorem. Moreover, we calculate the random sample mean S_n at each simulation cycle n where $0 \le n \le N$. Consequently, at each instance of n, we compute all the novel parameters of the complex probability paradigm (*CPP*) and *CLT* which are: P_r, P_m, P_m / i, *DOK*, *Chf*, *MChf*, *Pc*, and *Z*. After reaching the boundary value *N* for the simulation we exit the loop and draw all the corresponding parameters. This will help us greatly to prove, to quantify, and to illustrate all the functions of the original model and to do as well prognosis. Knowing that this methodology will be applied throughout the whole following section dedicated for simulations.

IX. The Simulation of the New Paradigm

Let us consider thereafter some stochastic distributions and theorems to simulate the Central Limit Theorem and to draw, to visualize, as well as to quantify all the *CPP* and prognostic parameters related to it. Note that all the numerical values found in the simulations of the new paradigm for any sample size and simulation cycles *n* were computed using the MATLAB version 2020 software. We have considered for this purpose a high capacity computer system: a workstation computer with parallel microprocessors, a 64-Bit operating system, and a 64-GB RAM.

9.1 The Simulation of the De Moivre–Laplace Theorem and CPP

The real convergence probability:

$$P_r(X) = P_{rob}(X \le x) = \sum_{k=0}^{x} \binom{n}{k} p^k q^{n-k}$$

= Cumulative distribution function (*CDF*) of the binomial distribution.

Where

x is a special instance or occurrence of the binomial random variable X

$0 \le k \le x : k = 0,1,2,\ldots,x$

$0 \le x \le n : x = 0,1,2,\ldots,n$

$p + q = 1, \quad p,q > 0$

and $\binom{n}{k} p^k q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k-np)^2}{2npq}}$ if $n \to +\infty$

with

$E(X) = \mu = np$, $\mathrm{Var}(X) = \sigma^2 = npq$, and $\text{Std. Deviation}(X) = \sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{npq}$

We have $0 \le X \le n$ where $X = 0$ corresponds to the instant before the beginning of the random experiment when $P_r(X \le 0) = \sum_{k=0}^{x=0} \binom{n}{k} p^k q^{n-k} = 0$, and $X = n$ corresponds to the instant at the end of the random binomial experiment and simulation when:

$$P_r(X \le n) = \sum_{k=0}^{x=n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1^n = 1 \text{ by the binomial theorem.}$$

The imaginary complementary divergence probability:

$$P_m(X) = i\left[1 - P_{rob}(X \le x)\right] = i\left[1 - \sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right] = iP_{rob}(X > x) = i\sum_{k=x+1}^{n}\binom{n}{k}p^k q^{n-k}$$

The real complementary divergence probability:

$$P_m(X)/i = 1 - P_{rob}(X \le x) = 1 - \sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k} = P_{rob}(X > x) = \sum_{k=x+1}^{n}\binom{n}{k}p^k q^{n-k}$$

The complex probability and random vector:

$$Z(X) = P_r(X) + P_m(X) = \sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k} + i\left[1 - \sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right]$$
$$= \sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k} + i\sum_{k=x+1}^{n}\binom{n}{k}p^k q^{n-k}$$

The Degree of Our Knowledge:

$$DOK(X) = |Z(X)|^2 = P_r^2(X) + [P_m(X)/i]^2 = \left[\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right]^2 + \left[1 - \sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right]^2$$
$$= 1 + 2iP_r(X)P_m(X) = 1 - 2P_r(X)\left[1 - P_r(X)\right] = 1 - 2P_r(X) + 2P_r^2(X)$$
$$= 1 - 2\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k} + 2\left[\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right]^2$$

$DOK(X)$ is equal to 1 when $P_r(X) = P_r(X \le 0) = 0$ and when $P_r(X) = P_r(X \le n) = 1$

The Chaotic Factor:

$$Chf(X) = 2iP_r(X)P_m(X) = -2P_r(X)\left[1 - P_r(X)\right] = -2P_r(X) + 2P_r^2(X)$$
$$= -2\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k} + 2\left[\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right]^2$$

$Chf(X)$ is null when $P_r(X) = P_r(X \le 0) = 0$ and when $P_r(X) = P_r(X \le n) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(X) = |Chf(X)| = -2iP_r(X)P_m(X) = 2P_r(X)\left[1 - P_r(X)\right] = 2P_r(X) - 2P_r^2(X)$$
$$= 2\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k} - 2\left[\sum_{k=0}^{x}\binom{n}{k}p^k q^{n-k}\right]^2$$

$MChf(X)$ is null when $P_r(X) = P_r(X \le 0) = 0$ and when $P_r(X) = P_r(X \le n) = 1$.

At any value of the random variable *X*: $0 \le \forall X \le n$, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(X) = [P_r(X) + P_m(X)/i]^2 = |Z(X)|^2 - 2iP_r(X)P_m(X)$$
$$= DOK(X) - Chf(X)$$
$$= DOK(X) + MChf(X)$$
$$= 1$$

then,

$Pc^2(X) = [P_r(X) + P_m(X)/i]^2 = \{P_r(X) + [1 - P_r(X)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(X) = 1$ always.

Hence, the prediction of the convergence probabilities of the stochastic experiments in the set $\mathcal{C}$ is permanently certain.

In the simulations, we take $p = q = 0.5$ and we have the following binomial distribution characteristics for the different values of n considered:

For $n = 8$, $\mu = 8 \times 0.5 = 4$, $\sigma^2 = 8 \times 0.5 \times 0.5 = 2 \Rightarrow \sigma = \sqrt{2} = 1.41421\ldots$

For $n = 12$, $\mu = 12 \times 0.5 = 6$, $\sigma^2 = 12 \times 0.5 \times 0.5 = 3 \Rightarrow \sigma = \sqrt{3} = 1.73205\ldots$

For $n = 16$, $\mu = 16 \times 0.5 = 8$, $\sigma^2 = 16 \times 0.5 \times 0.5 = 4 \Rightarrow \sigma = \sqrt{4} = 2$

For $n = 32$, $\mu = 32 \times 0.5 = 16$, $\sigma^2 = 32 \times 0.5 \times 0.5 = 8 \Rightarrow \sigma = \sqrt{8} = 2.82842\ldots$

For $n = 50$, $\mu = 50 \times 0.5 = 25$, $\sigma^2 = 50 \times 0.5 \times 0.5 = 12.5 \Rightarrow \sigma = \sqrt{12.5} = 3.53553\ldots$

For $n = 100$, $\mu = 100 \times 0.5 = 50$, $\sigma^2 = 100 \times 0.5 \times 0.5 = 25 \Rightarrow \sigma = \sqrt{25} = 5$

For $n = 10^6$, $\mu = 10^6 \times 0.5 = 500000$, $\sigma^2 = 10^6 \times 0.5 \times 0.5 = 250000 \Rightarrow \sigma = \sqrt{250000} = 500$

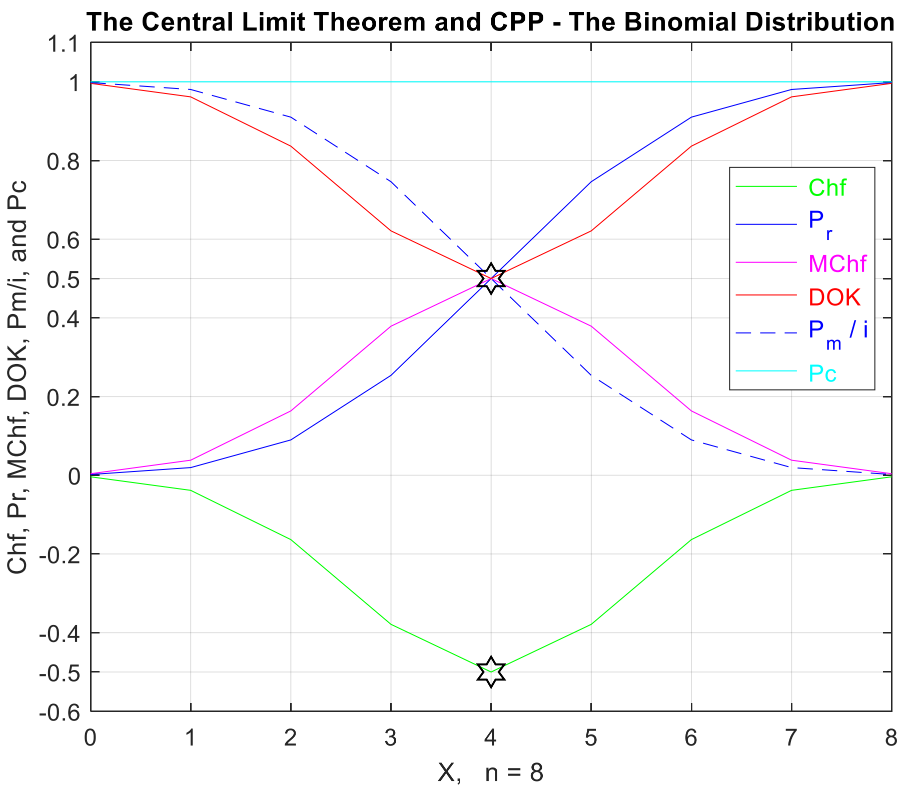


Figure 6: The De Moivre-Laplace Theorem and *CPP* for a sample of size $n = 8$

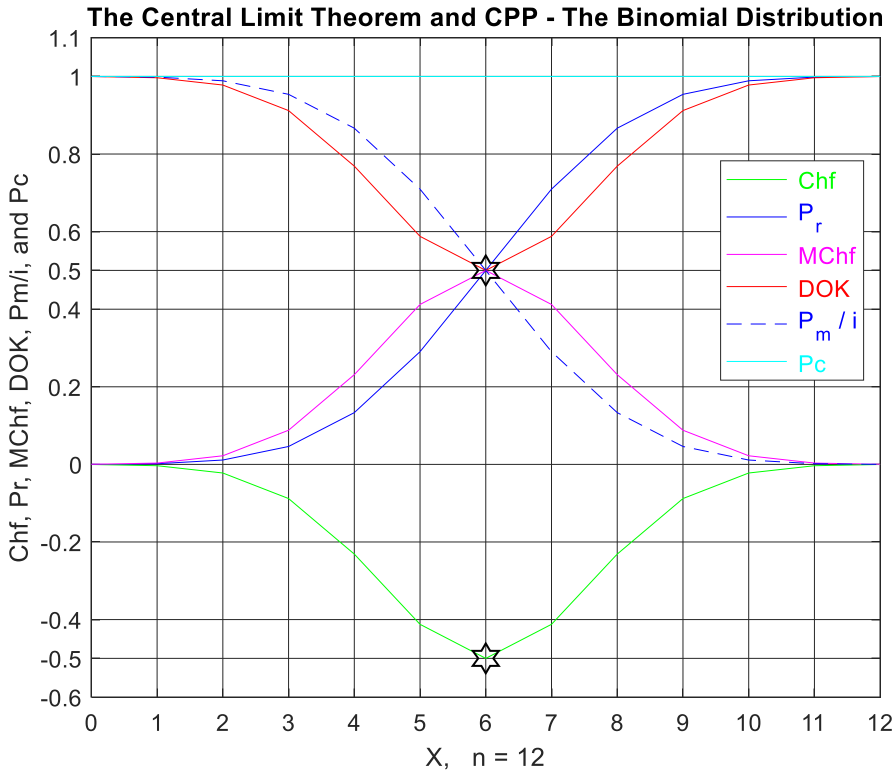


Figure 7: The De Moivre-Laplace Theorem and *CPP* for a sample of size $n = 12$

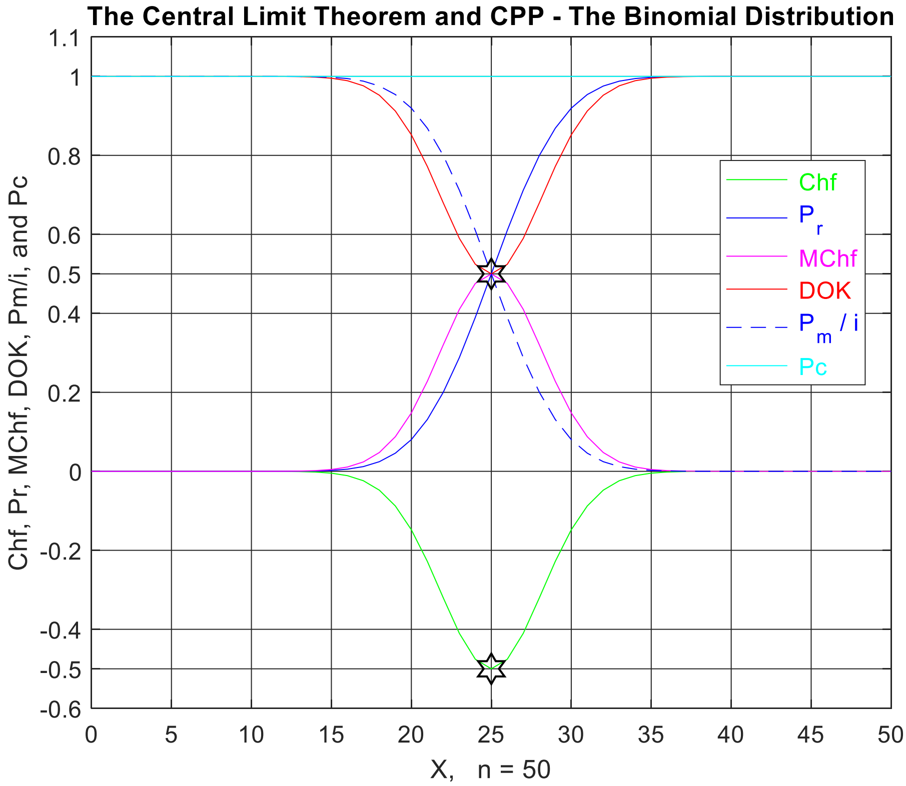


Figure 8: The De Moivre-Laplace Theorem and *CPP* for a sample of size $n = 50$

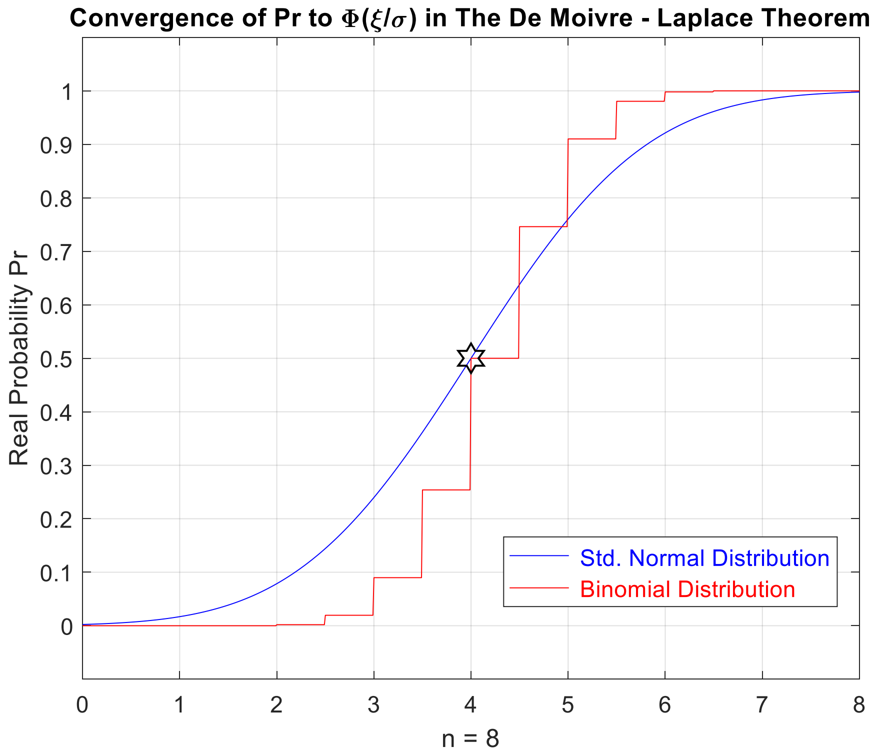


Figure 9: The increasing convergence of the binomial distribution to the std. normal distribution for a sample of size $n = 8$

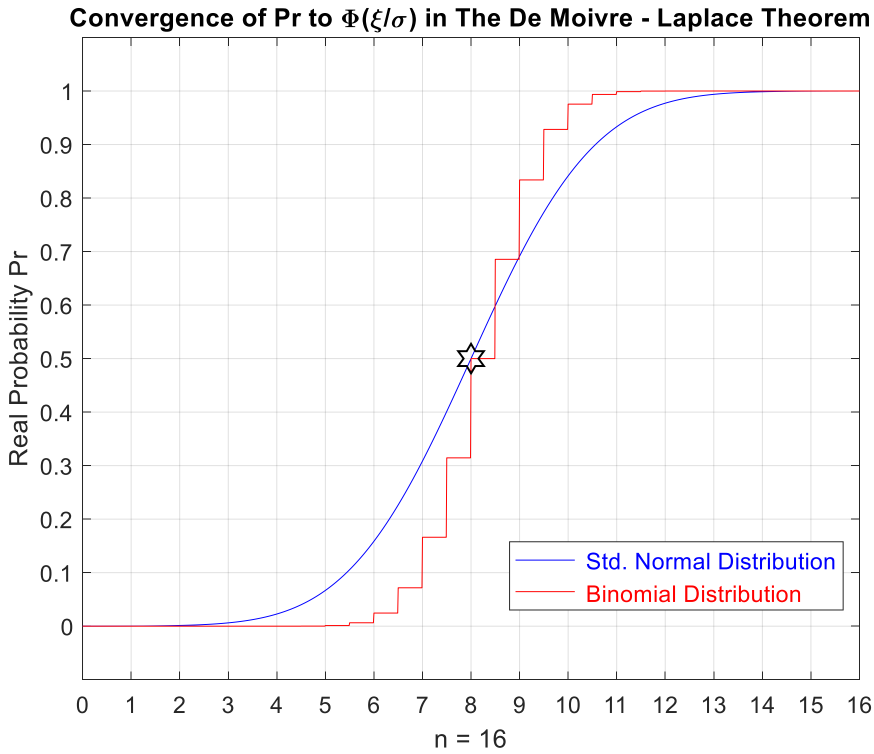


Figure 10: The increasing convergence of the binomial distribution to the std. normal distribution for a sample of size $n = 16$

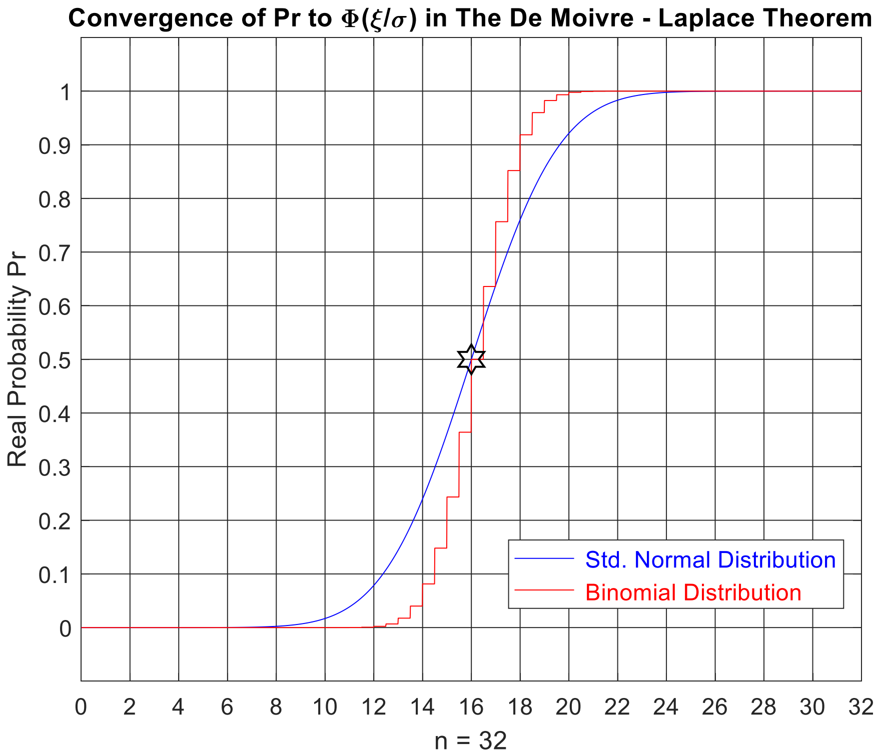


Figure 11: The increasing convergence of the binomial distribution to the std. normal distribution for a sample of size $n = 32$

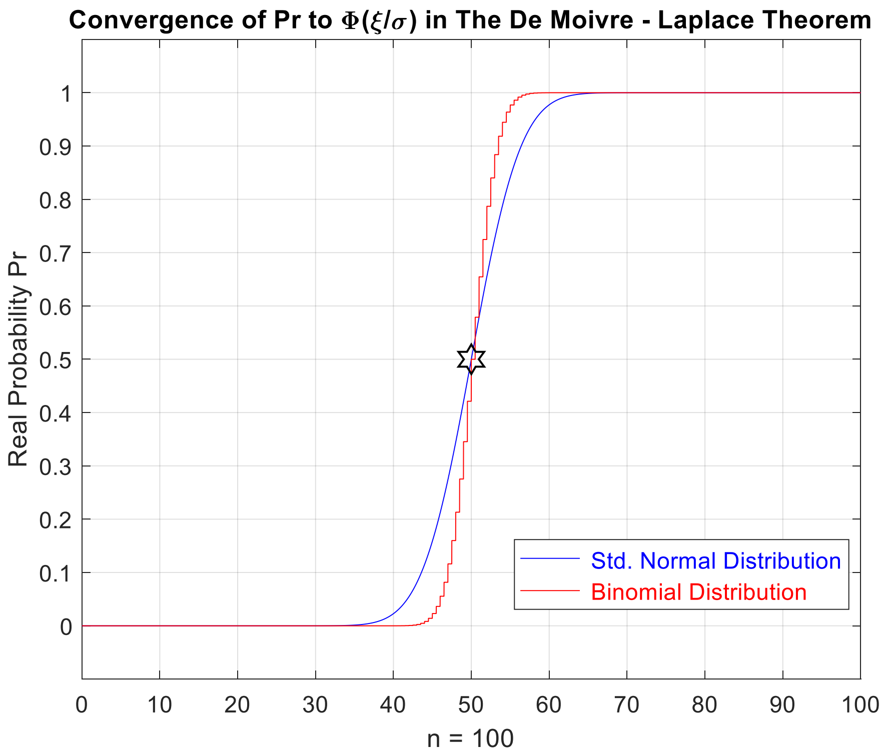


Figure 12: The increasing convergence of the binomial distribution to the std. normal distribution for a sample of size $n = 100$

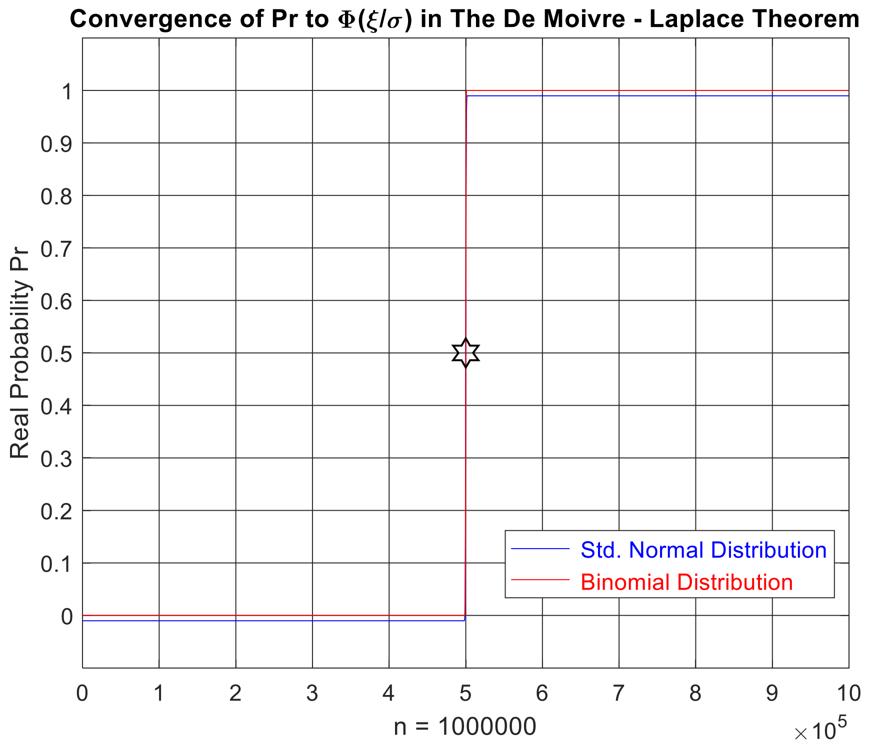


Figure 13: The increasing convergence of the binomial distribution to the std. normal distribution for a sample of size n = 1000000

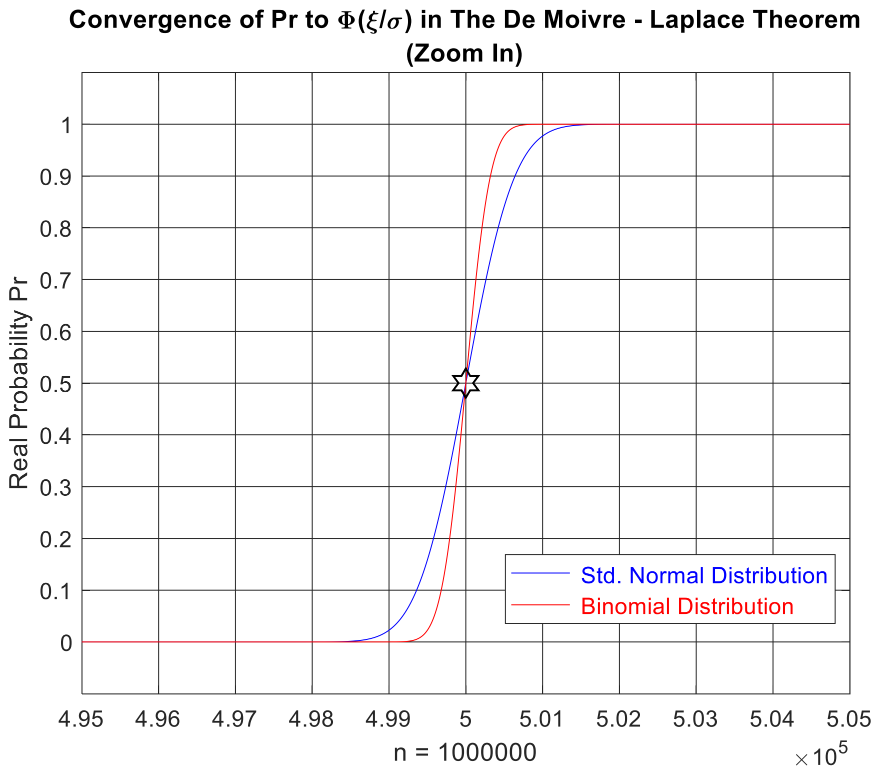


Figure 14: The increasing convergence of the binomial distribution to the std. normal distribution for a sample of size n = 1000000 (Zoom In)

9.1.1 The Simulations Interpretation

After considering the De Moivre–Laplace theorem, hence the binomial distribution, we can deduce a value of $P_r(X)$ for each value of the random variable X and for each value of the random sample size n. Figures 6, 7, and 8 illustrate all the new prognostic model functions and prove all the mathematical derivations. We have computed and ploted for a special set of $P_r(X)$ all the *CPP* parameters and components and which are: *Chf*(X), *MChf*(X), *DOK*(X), *Pc*(X), $P_m(X)/i$, and showed how to calculate the correpsonding $Z(X)$. This is achieved with an increasing value of n by taking into consideration the cases $n = 8$, 12, and 50 to illustrate the paradigm.

Furthermore, as it was verified and demonstrated in the original model, when $n = 0$ (before the random simulation beginning) and at n (when the simulation converges) then the degree of our knowledge (*DOK*) is 1 and the chaotic factor (*Chf* and *MChf*) is 0 since the stochastic effects and fluctuations have either not started yet or they have finished their task on the random experiment and simulation. We note from these figures that the *DOK* is maximum (*DOK* = 1) when absolute value of *Chf* which is *MChf* is minimum (*MChf* = 0), that means when the magnitude of the chaotic factor (*MChf*) decreases our certain knowledge (*DOK*) increases. Subsequently, *MChf* begins to grow during the simulation due to the intrinsic conditions thus leading to a decrease in *DOK* until they both reach 0.5 at $n/2$ in all possible cases. During the course of the nondeterministic and stochastic experiment ($n > 0$) we have: $0.5 \le DOK < 1$, $-0.5 \le Chf < 0$, and $0 < MChf \le 0.5$. The real cumulative convergence probability P_r and the real cumulative complementary divergence probability P_m/i will meet with *DOK* and *MChf* also at the point ($n/2$, 0.5) in all possible cases also. With the growth of X, the *Chf* and *MChf* return to zero and the *DOK* returns to 1 where we attain the total convergence of the binomial distribution to a normal distribution as predicted by De Moivre–Laplace theorem and *CLT* ($P_r = 1$) as $n >> 1$ or $n \to +\infty$. At this last point, and for large n, convergence here is definite since $P_r(X) = 1$ with $Pc(X) = 1$ permanently, so the logical consequence of the value *DOK* = 1 follows.

We note that $n/2$ corresponds to $X_{Median} = X_{Mean} = X_{Mode}$ of the distribution and which are at the middle of the simulations since the binomial and normal distributions considered here are totally symmetric, therefore the corresponding graphs are perfectly symmetric.

Moreover, at each value of X and n and during this entire process, we can predict with certainty all the *CPP* parameters in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ with *Pc* preserved as equal to one through a continuous compensation between *DOK* and *Chf* since $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$ in the *CPP*. This compensation is from the instant $n = 0$ (at the beginning of the random sampling and simulation) where $P_r(X) = 0$ until the instant of convergence n (at the end of the random sampling and simulation) where $P_r(X) = 1$. That means also that the simulation which looked to be random and nondeterministic in the set $\mathcal{R}$ is now deterministic and certain in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding the contributions of $\mathcal{M}$ to the experiment happening in $\mathcal{R}$ and thus after removing and subtracting the chaotic factor from the degree of our knowledge in the equation above.

Additionally, Figures 9 to 14 show the increasing convergence probability of the binomial distribution to the normal (or the standard normal = $\Phi(\xi/\sigma)$) distribution with the increasing value of n by considering the values $n = 8$, 16, 32, 100, and 1000000, just as predicted by De Moivre–Laplace theorem which is a special case of *CLT* that considers the binomial distribution for the random variable X.

The real convergence probability:

$$P_r(X) = P_{rob}(X \le x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

= Cumulative distribution function (*CDF*) of the Poisson distribution.

Where
x is a special instance or occurrence of the Poisson random variable X
$0 \le k \le x : k = 0,1,2,\ldots,x$
$0 \le x < +\infty : x = 0,1,2,\ldots,+\infty$

For sufficiently large values of n and with a sufficiently small values of $\lambda = np$ we have:
$F_{\text{Binomial}}(k;n,p) \simeq F_{\text{Poisson}}(k;\lambda = np)$

Therefore, $\binom{n}{k} p^k q^{n-k} \simeq \frac{\lambda^k e^{-\lambda}}{k!}$

For sufficiently large values of λ and with an appropriate continuity correction we have:
$F_{\text{Poisson}}(x;\lambda = np) \simeq F_{\text{Normal}}(x;\mu = \lambda, \sigma^2 = \lambda)$

Therefore, $\frac{\lambda^k e^{-\lambda}}{k!} \simeq \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(k-\lambda)^2}{2\lambda}}$

And we have: $E(X) = \mu = np = \lambda$, $\text{Var}(X) = \sigma^2 = \lambda$, and
$\text{Std. Deviation}(X) = \sigma = \sqrt{\text{Var}(X)} = \sqrt{\lambda}$

We have $0 \le X < +\infty$ where $X = 0$ corresponds to the instant before the beginning of the random experiment and simulation when $P_r(X \le 0) = \sum_{k=0}^{x=0} \frac{\lambda^k e^{-\lambda}}{k!} = 0$, and $X >> 1$ (For large x or for $x \to +\infty$) corresponds to the instant at the end of the random Poisson simulation when:
$P_r(X >> 1) = \sum_{k=0}^{x>>1} \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{+\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \times e^{\lambda} = e^0 = 1$ after using the series properties from calculus.

The imaginary complementary divergence probability:

$$P_m(X) = i\left[1 - P_{rob}(X \le x)\right] = i\left[1 - \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right] = iP_{rob}(X > x) = i\sum_{k=x+1}^{+\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$

The real complementary divergence probability:

$$P_m(X)/i = 1 - P_{rob}(X \le x) = 1 - \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} = P_{rob}(X > x) = \sum_{k=x+1}^{+\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$

The complex probability and random vector:

$$Z(X) = P_r(X) + P_m(X) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} + i\left[1 - \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right]$$

$$= \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} + i\sum_{k=x+1}^{+\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$

The Degree of Our Knowledge:

$$DOK(X) = |Z(X)|^2 = P_r^2(X) + [P_m(X)/i]^2 = \left[\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right]^2 + \left[1 - \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right]^2$$

$$= 1 + 2iP_r(X)P_m(X) = 1 - 2P_r(X)[1 - P_r(X)] = 1 - 2P_r(X) + 2P_r^2(X)$$

$$= 1 - 2\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} + 2\left[\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right]^2$$

$DOK(X)$ is equal to 1 when $P_r(X) = P_r(X \le 0) = 0$ and when $P_r(X) = P_r(X >> 1) = 1$

The Chaotic Factor:

$$Chf(X) = 2iP_r(X)P_m(X) = -2P_r(X)[1 - P_r(X)] = -2P_r(X) + 2P_r^2(X)$$

$$= -2\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} + 2\left[\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right]^2$$

$Chf(X)$ is null when $P_r(X) = P_r(X \le 0) = 0$ and when $P_r(X) = P_r(X >> 1) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(X) = |Chf(X)| = -2iP_r(X)P_m(X) = 2P_r(X)[1 - P_r(X)] = 2P_r(X) - 2P_r^2(X)$$

$$= 2\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} - 2\left[\sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}\right]^2$$

$MChf(X)$ is null when $P_r(X) = P_r(X \le 0) = 0$ and when $P_r(X) = P_r(X >> 1) = 1$.

At any value of the random variable *X*: $0 \le \forall X < +\infty$, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(X) = [P_r(X) + P_m(X)/i]^2 = |Z(X)|^2 - 2iP_r(X)P_m(X)$$

$$= DOK(X) - Chf(X)$$

$$= DOK(X) + MChf(X)$$

$$= 1$$

then,

$Pc^2(X) = [P_r(X) + P_m(X)/i]^2 = \{P_r(X) + [1 - P_r(X)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(X) = 1$ always.

Hence, the prediction of the convergence probabilities of the stochastic experiments in the set $\mathcal{C}$ is permanently certain.

In the simulations, we have considered the following Poisson distribution characteristics:

$$\mu = \lambda = 6.7,\ \sigma^2 = \lambda = 6.7 \Rightarrow \sigma = \sqrt{\lambda} = \sqrt{6.7} = 2.58843\ldots$$

$$\mu = \lambda = 10.7,\ \sigma^2 = \lambda = 10.7 \Rightarrow \sigma = \sqrt{\lambda} = \sqrt{10.7} = 3.27108\ldots$$

$$\mu = \lambda = 35.7,\ \sigma^2 = \lambda = 35.7 \Rightarrow \sigma = \sqrt{\lambda} = \sqrt{35.7} = 5.97494\ldots$$

For *n* = 8, 16, 32, 100, 1000000, we have $\mu = \lambda = 6.7$.

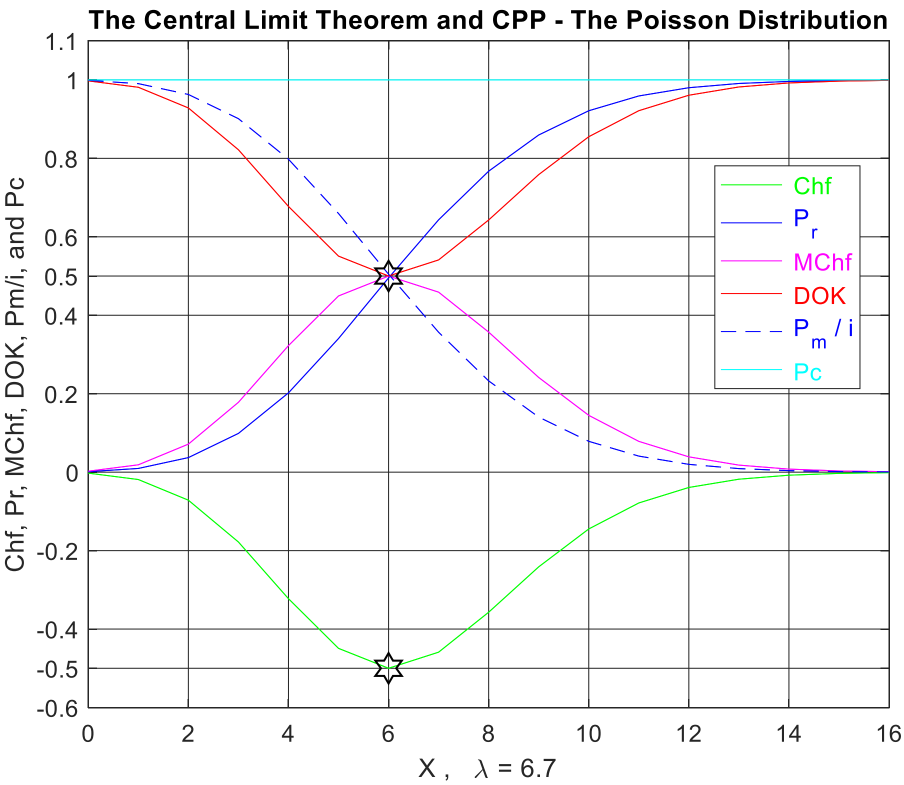


Figure 15: The Poisson Theorem and *CPP* for $\lambda = 6.7$

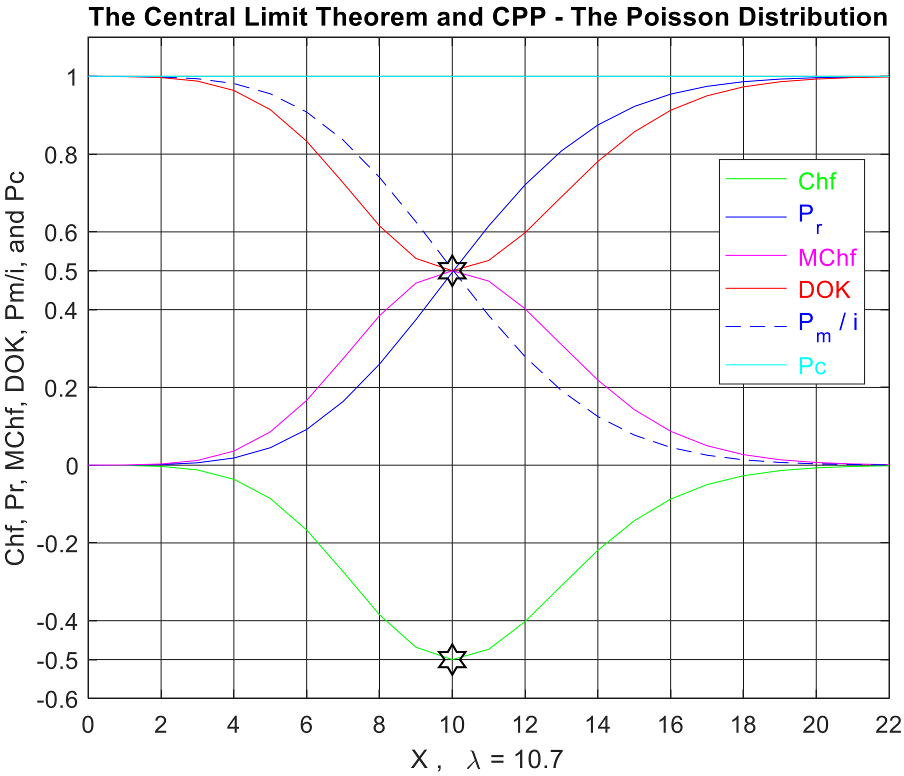


Figure 16: The Poisson Theorem and *CPP* for $\lambda = 10.7$

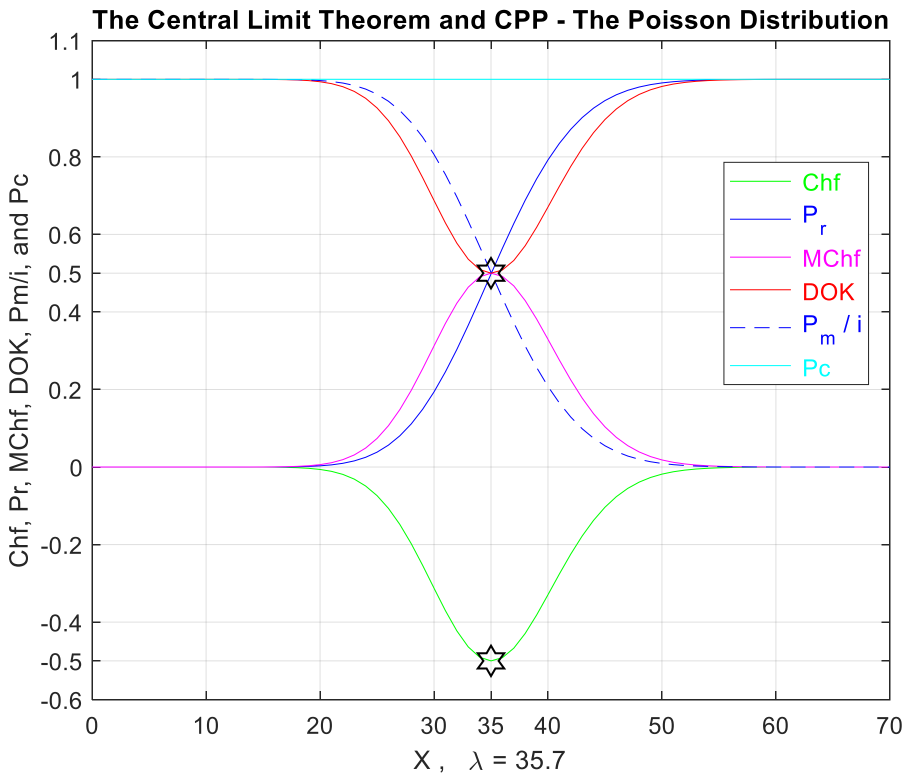


Figure 17: The Poisson Theorem and *CPP* for $\lambda = 35.7$

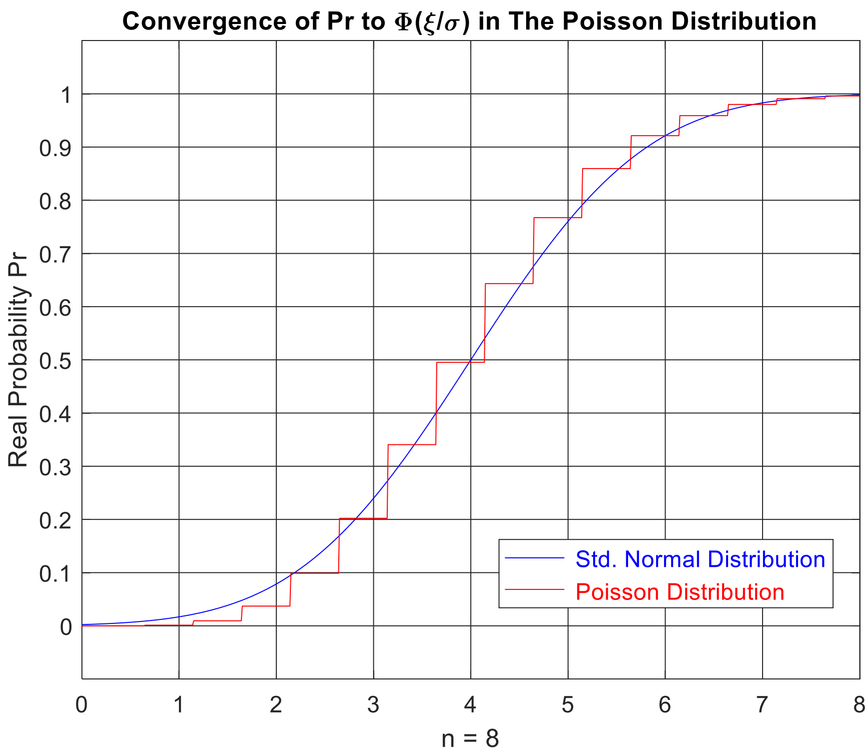


Figure 18: The increasing convergence of the Poisson distribution to the std. normal distribution for a sample of size n = 8 with $\lambda = 6.7$

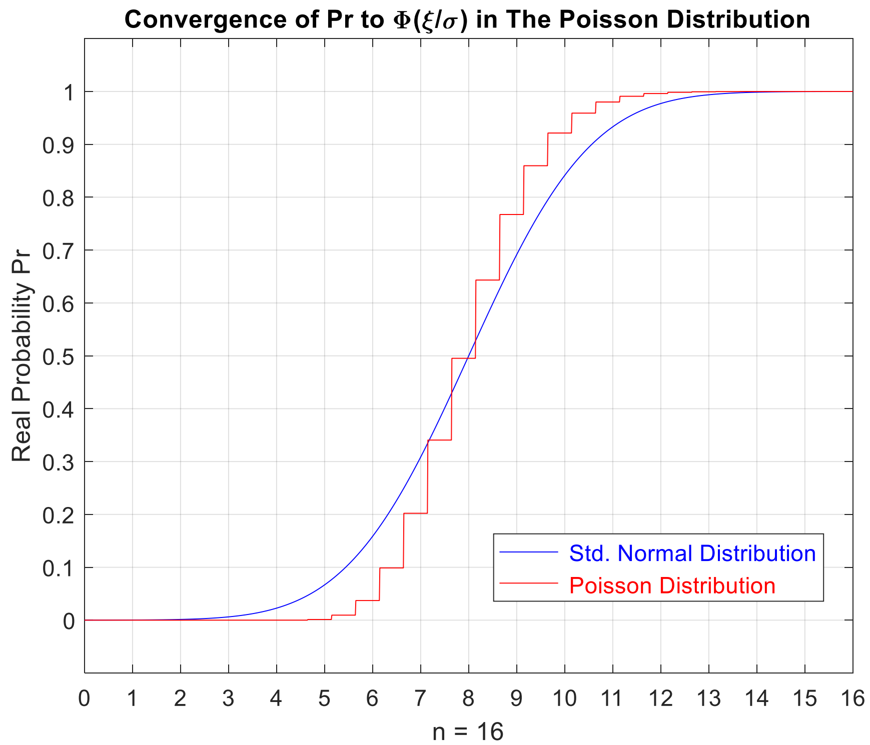


Figure 19: The increasing convergence of the Poisson distribution to the std. normal distribution for a sample of size n = 16 with $\lambda = 6.7$

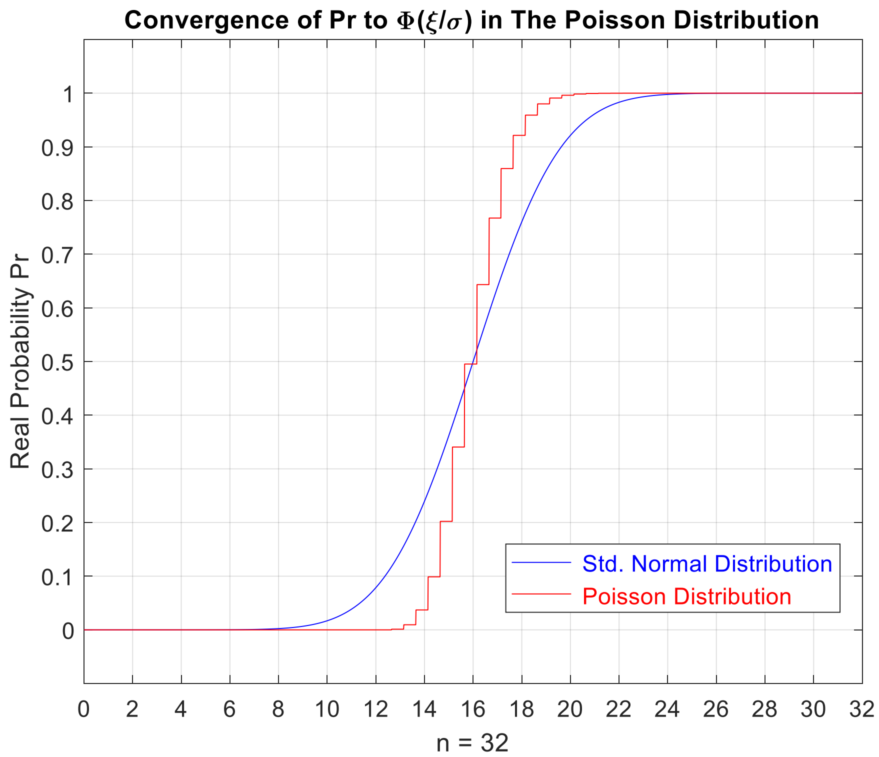


Figure 20: The increasing convergence of the Poisson distribution to the std. normal distribution for a sample of size n = 32 with $\lambda = 6.7$

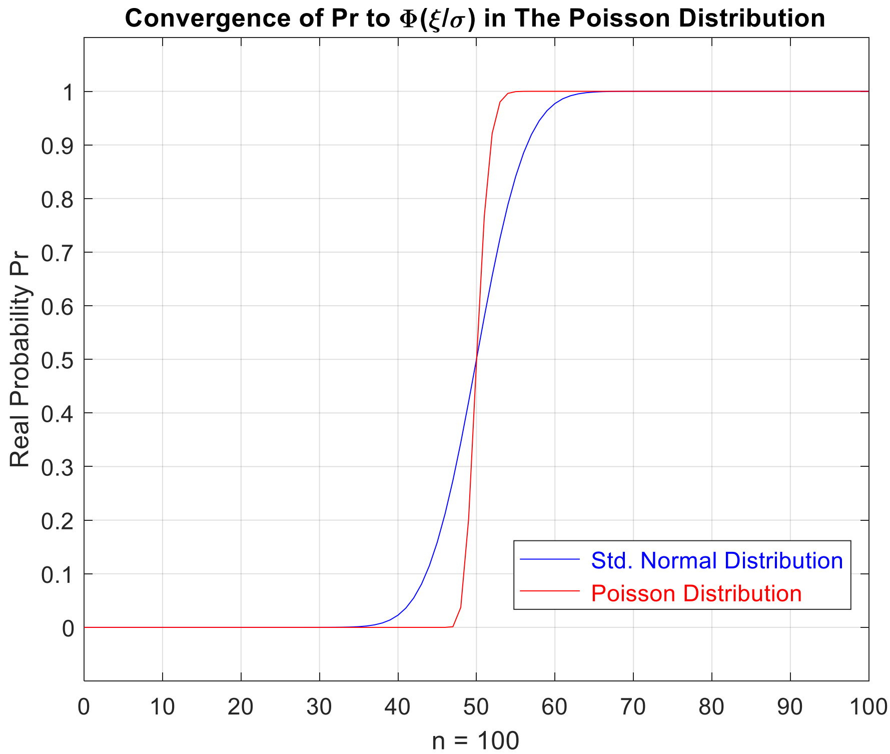


Figure 21: The increasing convergence of the Poisson distribution to the std. normal distribution for a sample of size n = 100 with $\lambda = 6.7$

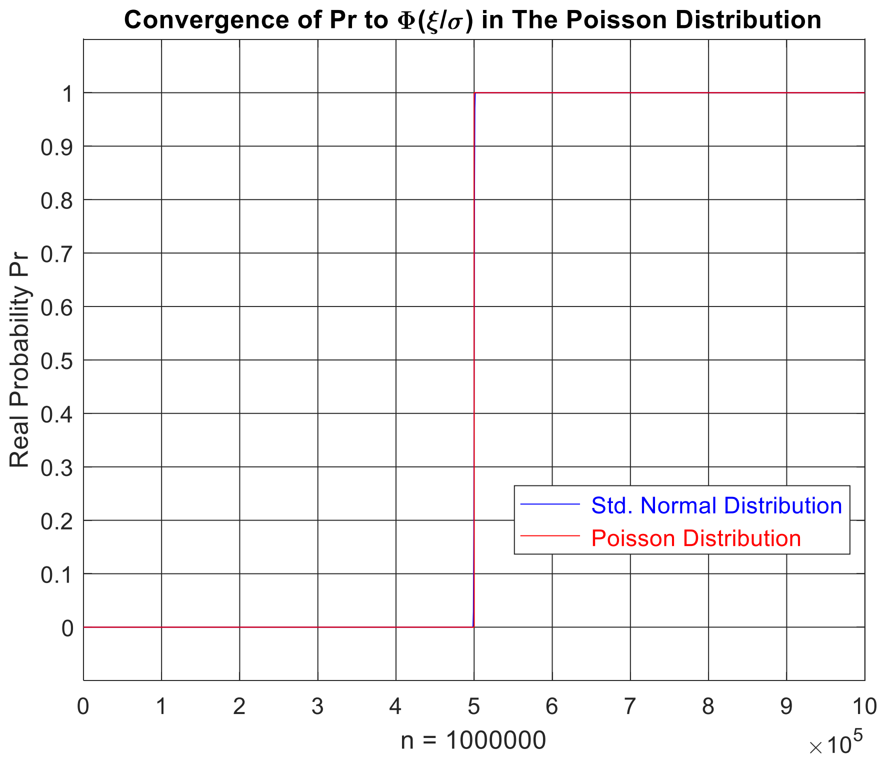


Figure 22: The increasing convergence of the Poisson distribution to the std. normal distribution for a sample of size n = 1000000 with $\lambda = 6.7$

9.2.1 The Simulations Interpretation

After considering now the Poisson distribution, we can deduce a value of $P_r(X)$ for each value of the random variable X, for each value of λ, and for each value of the random sample size n. Figures 15, 16, and 17 illustrate all the new prognostic model functions and prove all the mathematical derivations. We have computed and drawn for a special set of $P_r(X)$ all the *CPP* parameters and components and which are: *Chf*(X), *MChf*(X), *DOK*(X), *Pc*(X), $P_m(X)/i$, and showed how to calculate the correpsonding $Z(X)$. This is achieved with the increasing value of λ by taking into consideration the cases λ = 6.7, 10.7, and 35.7 to illustrate the paradigm.

Furthermore, as it was proved and confirmed in the original model, when $n = 0$ (before the random simulation beginning) and at n (when the simulation converges) then the degree of our knowledge (*DOK*) is 1 and the chaotic factor (*Chf* and *MChf*) is 0 since the stochastic aspects and fluctuations have either not begun yet or they have completed their task on the random phenomenon and simulation. We note from these figures that the *DOK* is maximum (*DOK* = 1) when absolute value of *Chf* which is *MChf* is minimum (*MChf* = 0), that means when the magnitude of the chaotic factor (*MChf*) diminishes our certain knowledge (*DOK*) grows. Subsequently, *MChf* begins to increase during the simulation due to the intrinsic conditions thus leading to a decrease in *DOK* until they both reach 0.5 at $\lfloor \lambda \rfloor = \text{Floor}(\lambda)$ in all these cases. During the course of the nondeterministic and stochastic experiment ($n > 0$) we have: $0.5 \le DOK < 1$, $-0.5 \le Chf < 0$, and $0 < MChf \le 0.5$. The real cumulative convergence probability P_r and the real cumulative complementary divergence probability P_m/i will meet with *DOK* and *MChf* also at the point $(X_{Median} = X_{Mode} = \lfloor \lambda \rfloor, 0.5)$ in all these cases also. With the growth of X, the *Chf* and *MChf* return to zero and the *DOK* returns to 1 where we attain the total convergence of the Poisson distribution to a normal distribution as predicted by the Poisson theorem and *CLT* ($P_r = 1$) as $\lambda >> 1$, $n >> 1$ or $n \to +\infty$. At this last point, and for large λ and n, convergence here is definite since $P_r(X) = 1$ with $Pc(X) = 1$ permanently, so the logical consequence of the value $DOK = 1$ follows.

We note that $\lfloor \lambda \rfloor$ corresponds to X_{Mode} of the distribution where $X_{Mean} = \bar{X} = E(X) = \lambda$ and $X_{Median} \simeq \lfloor \lambda + 1/3 - 0.02/\lambda \rfloor$ and which are not at the middle of the simulations since the Poisson distribution considered is not symmetric, therefore the corresponding graphs considered here are skewed to the right or positively skewed before the convergence of the Poisson distribution to a normal distribution when it becomes perfectly symmetric.

Moreover, at each value of X, λ, and n and during this entire process, we can predict with certainty all the *CPP* parameters in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ with *Pc* preserved as equal to one through a continuous compensation between *DOK* and *Chf* since $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$ in the *CPP*. This compensation is from the instant $n = 0$ (at the beginning of the random sampling and simulation) where $P_r(X) = 0$ until the instant of convergence n (at the end of the random sampling and simulation) where $P_r(X) = 1$. That means also that the simulation which seemed to be random and nondeterministic in the set $\mathcal{R}$ is now deterministic and certain in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding the contributions of $\mathcal{M}$ to the experiment occurring in $\mathcal{R}$ and thus after eliminating and subtracting the chaotic factor from the degree of our knowledge in the equation above.

Additionally, Figures 18 to 22 show the increasing convergence probability of the Poisson distribution to the normal (or the standard normal = $\Phi(\xi/\sigma)$) distribution with the increasing value of n by considering the values n = 8, 16, 32, 100, and 1000000, just as predicted by the Poisson theorem which is a special case of *CLT* that considers the Poisson distribution for the random variable *X*.

9.3 The Simulation of the CLT

9.3.1 The Simulation of the CLT and CPP

The real convergence probability is:

$$P_r(\Xi) = P_{rob}(\Xi \le \xi / \sigma) = P_{rob}\left[\frac{\sqrt{n}(S_n - \mu)}{\sigma} \le \frac{\xi}{\sigma}\right]$$

= Cumulative distribution function (*CDF*) of the S_n distribution.

Where

$$\Xi = \frac{\sqrt{n}(S_n - \mu)}{\sigma} = \frac{S_n - \mu}{\sigma / \sqrt{n}}$$

And ξ is a special instance or occurrence of the random variable Ξ and it can be any real number.

The sample mean S_n of size n is taken here from a population following a binomial distribution having the following characteristics:

$\mu = np$, $\text{Variance} = \sigma^2 = npq$, and $\text{Std. Deviation} = \sigma = \sqrt{\text{Variance}} = \sqrt{npq}$

We note that $\sigma / \sqrt{n}$ is called the standard error of the sample mean S_n.

We have:

As n approaches infinity, the random variables $\sqrt{n}(S_n - \mu)$ converge in distribution to a normal distribution $N(0, \sigma^2)$, so: $\sqrt{n}(S_n - \mu) \to N(0, \sigma^2)$.

Or we can write for every real number ξ:

$$\lim_{n \to +\infty} P_r(\Xi) = \lim_{n \to +\infty} P_{rob}(\Xi \le \xi / \sigma) = \lim_{n \to +\infty} P_{rob}\left[\frac{\sqrt{n}(S_n - \mu)}{\sigma} \le \frac{\xi}{\sigma}\right] = \Phi\left(\frac{\xi}{\sigma}\right)$$

where $\Phi(\xi)$ is the standard normal *CDF* evaluated at ξ.

Accordingly, and since the distribution of Ξ is centered and reduced, then for large n or for $n \to +\infty$ we have:

$E(\Xi) = 0$, $\text{Var}(\Xi) = 1$, and $\text{Std. Deviation}(\Xi) = \sqrt{\text{Var}(\Xi)} = \sqrt{1} = 1$

We have $-\infty < \Xi < +\infty$ where n = 0 corresponds to the instant before the beginning of the random sampling when $P_r(\Xi) = 0$, and n corresponds to the instant at the end of the random sampling and simulation when $P_r(\Xi) = 1$.

The imaginary complementary divergence probability:

$$P_m(\Xi) = i\left[1 - P_{rob}(\Xi \le \xi / \sigma)\right] = i\left[1 - P_{rob}\left[\frac{\sqrt{n}(S_n - \mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]$$

$$= iP_{rob}(\Xi > \xi / \sigma) = iP_{rob}\left[\frac{\sqrt{n}(S_n - \mu)}{\sigma} > \frac{\xi}{\sigma}\right]$$

The real complementary divergence probability:

$$P_m(\Xi)/i = 1 - P_{rob}(\Xi \le \xi/\sigma) = 1 - P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]$$

$$= P_{rob}(\Xi > \xi/\sigma) = P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} > \frac{\xi}{\sigma}\right]$$

The complex probability and random vector:

$$Z(\Xi) = P_r(\Xi) + P_m(\Xi) = P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right] + i\left[1 - P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]$$

$$= P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right] + iP_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} > \frac{\xi}{\sigma}\right]$$

The Degree of Our Knowledge:

$$DOK(\Xi) = |Z(\Xi)|^2 = P_r^2(\Xi) + [P_m(\Xi)/i]^2$$

$$= \left[P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]^2 + \left[1 - P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]^2$$

$$= 1 + 2iP_r(\Xi)P_m(\Xi) = 1 - 2P_r(\Xi)[1 - P_r(\Xi)] = 1 - 2P_r(\Xi) + 2P_r^2(\Xi)$$

$$= 1 - 2P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right] + 2\left[P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]^2$$

$DOK(\Xi)$ is equal to 1 when $P_r(\Xi) = P_r(n=0) = 0$ and when $P_r(\Xi) = P_r(n) = 1$ that means at the end of the simulation.

The Chaotic Factor:

$$Chf(\Xi) = 2iP_r(\Xi)P_m(\Xi) = -2P_r(\Xi)[1 - P_r(\Xi)] = -2P_r(\Xi) + 2P_r^2(\Xi)$$

$$= -2P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right] + 2\left[P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]^2$$

$Chf(\Xi)$ is null when $P_r(\Xi) = P_r(n=0) = 0$ and when $P_r(\Xi) = P_r(n) = 1$ that means at the end of the simulation.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(\Xi) = |Chf(\Xi)| = -2iP_r(\Xi)P_m(\Xi) = 2P_r(\Xi)[1 - P_r(\Xi)] = 2P_r(\Xi) - 2P_r^2(\Xi)$$

$$= 2P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right] - 2\left[P_{rob}\left[\frac{\sqrt{n}(S_n-\mu)}{\sigma} \le \frac{\xi}{\sigma}\right]\right]^2$$

$MChf(\Xi)$ is null when $P_r(\Xi) = P_r(n=0) = 0$ and when $P_r(\Xi) = P_r(n) = 1$ that means at the end of the simulation.

At any value of the random variable $\Xi : -\infty < \forall \Xi < +\infty$ and for any value of the sample size *n*, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(\Xi) = [P_r(\Xi) + P_m(\Xi)/i]^2 = |Z(\Xi)|^2 - 2iP_r(\Xi)P_m(\Xi)$$
$$= DOK(\Xi) - Chf(\Xi)$$
$$= DOK(\Xi) + MChf(\Xi)$$
$$= 1$$

then,

$$Pc^2(\Xi) = [P_r(\Xi) + P_m(\Xi)/i]^2 = \{P_r(\Xi) + [1 - P_r(\Xi)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(\Xi) = 1 \text{ always.}$$

Hence, the prediction of the convergence probabilities of the stochastic experiments in the set $\mathcal{C}$ is permanently certain.

In the simulations, we take $p = q = 0.5$ and we have considered the following binomial distribution characteristics:

For $n = 4$, $\mu = 4 \times 0.5 = 2$, $\sigma^2 = 4 \times 0.5 \times 0.5 = 1 \Rightarrow \sigma = \sqrt{1} = 1$

For $n = 8$, $\mu = 8 \times 0.5 = 4$, $\sigma^2 = 8 \times 0.5 \times 0.5 = 2 \Rightarrow \sigma = \sqrt{2} = 1.41421\ldots$

For $n = 16$, $\mu = 16 \times 0.5 = 8$, $\sigma^2 = 16 \times 0.5 \times 0.5 = 4 \Rightarrow \sigma = \sqrt{4} = 2$

For $n = 30$, $\mu = 30 \times 0.5 = 15$, $\sigma^2 = 30 \times 0.5 \times 0.5 = 7.5 \Rightarrow \sigma = \sqrt{7.5} = 2.73861...$

For $n = 44$, $\mu = 44 \times 0.5 = 22$, $\sigma^2 = 44 \times 0.5 \times 0.5 = 11 \Rightarrow \sigma = \sqrt{11} = 3.31662\ldots$

For $n = 100$, $\mu = 100 \times 0.5 = 50$, $\sigma^2 = 100 \times 0.5 \times 0.5 = 25 \Rightarrow \sigma = \sqrt{25} = 5$

For $n = 1000$, $\mu = 1000 \times 0.5 = 500$, $\sigma^2 = 1000 \times 0.5 \times 0.5 = 250 \Rightarrow \sigma = \sqrt{250} = 15.81138...$

For $n = 10000$, $\mu = 10000 \times 0.5 = 5000$, $\sigma^2 = 10000 \times 0.5 \times 0.5 = 2500 \Rightarrow \sigma = \sqrt{2500} = 50$

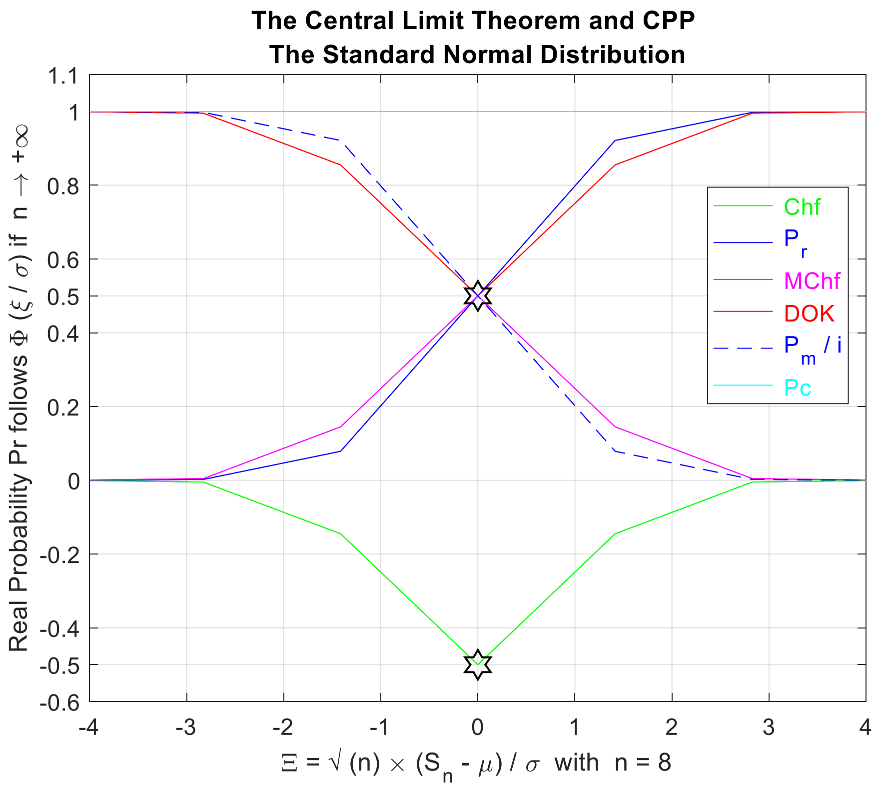


Figure 23: The random variable $\Xi = \sqrt{n}\left(S_n - \mu\right)/\sigma$ in *CLT* and *CPP* for *n* = 8

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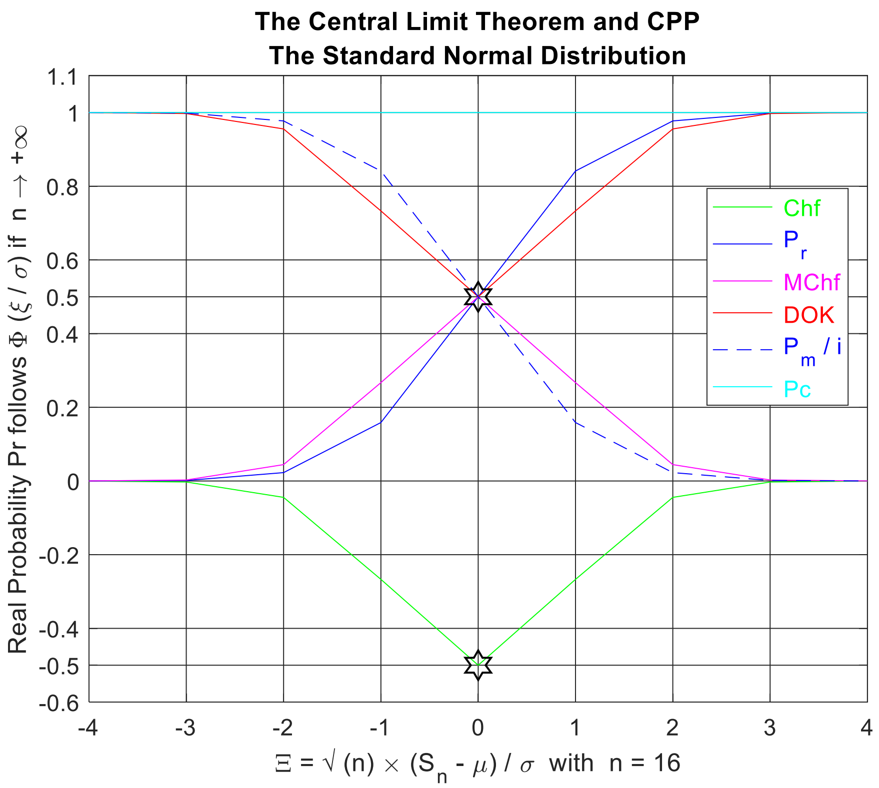

Figure 24: The random variable $\Xi = \sqrt{n}\left(S_n - \mu\right)/\sigma$ in *CLT* and *CPP* for $n = 16$

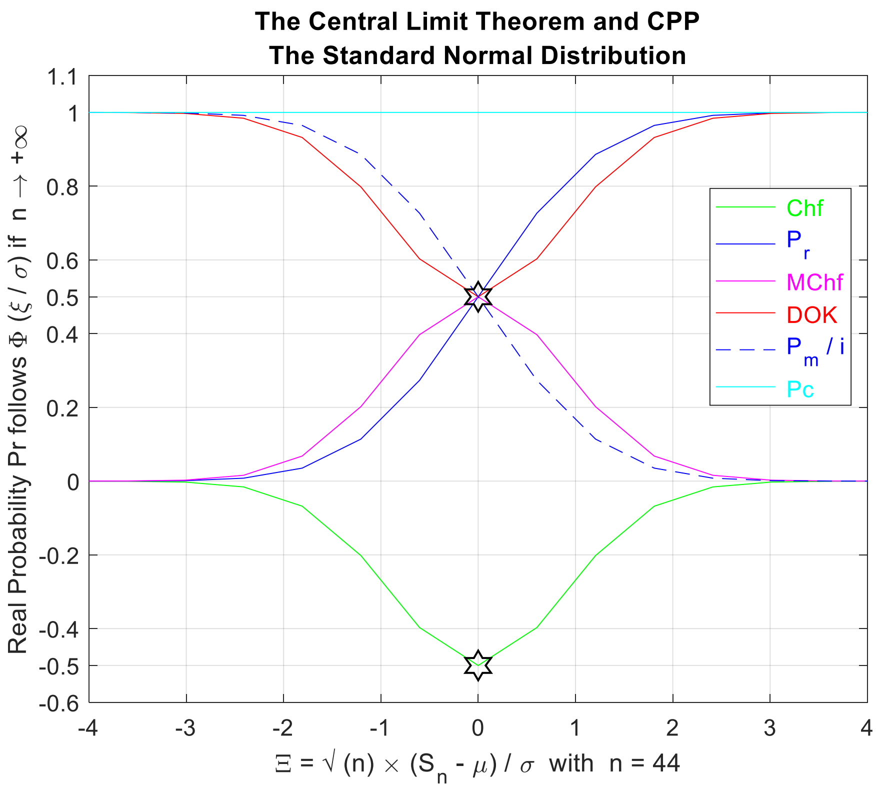

Figure 25: The random variable $\Xi = \sqrt{n}\left(S_n - \mu\right)/\sigma$ in *CLT* and *CPP* for $n = 44$

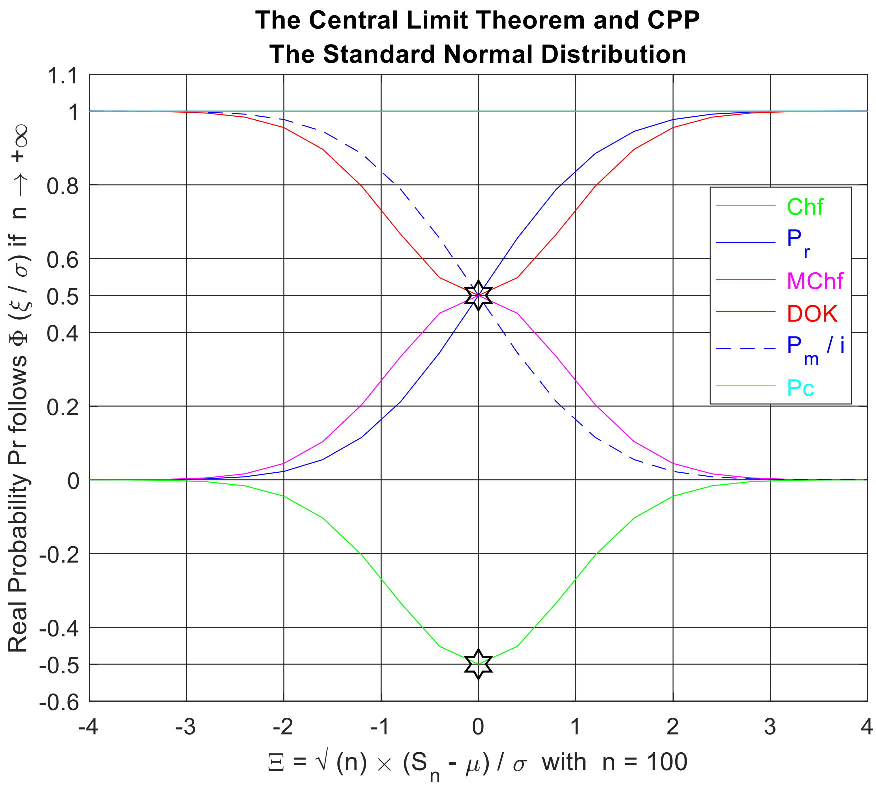

Figure 26: The random variable $\Xi = \sqrt{n}\left(S_n - \mu\right)/\sigma$ in *CLT* and *CPP* for $n = 100$

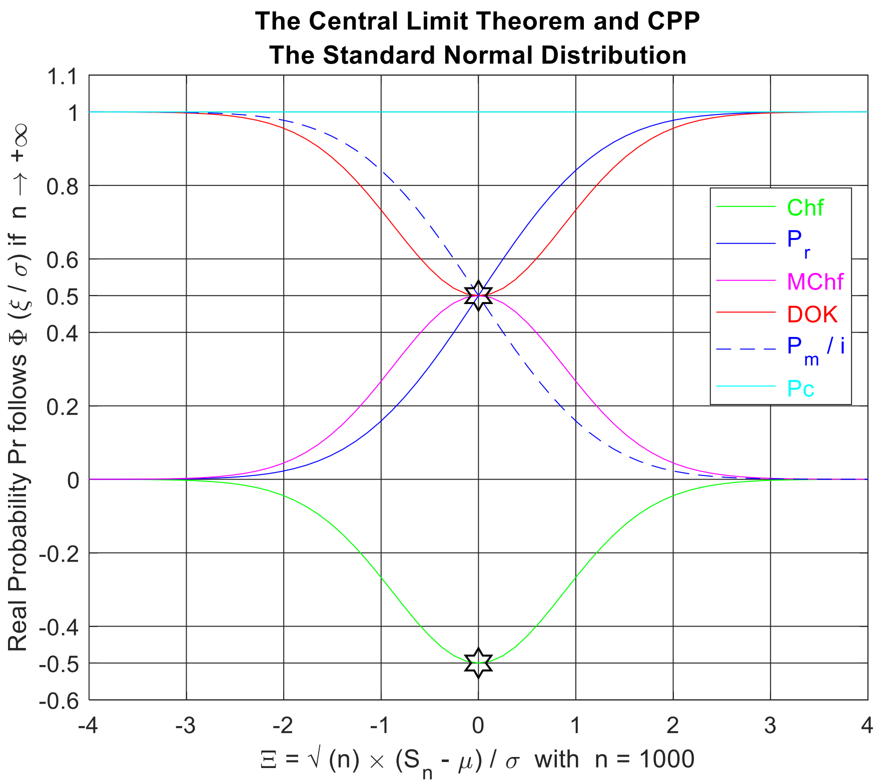

Figure 27: The random variable $\Xi = \sqrt{n}\left(S_n - \mu\right)/\sigma$ in *CLT* and *CPP* for $n = 1000$

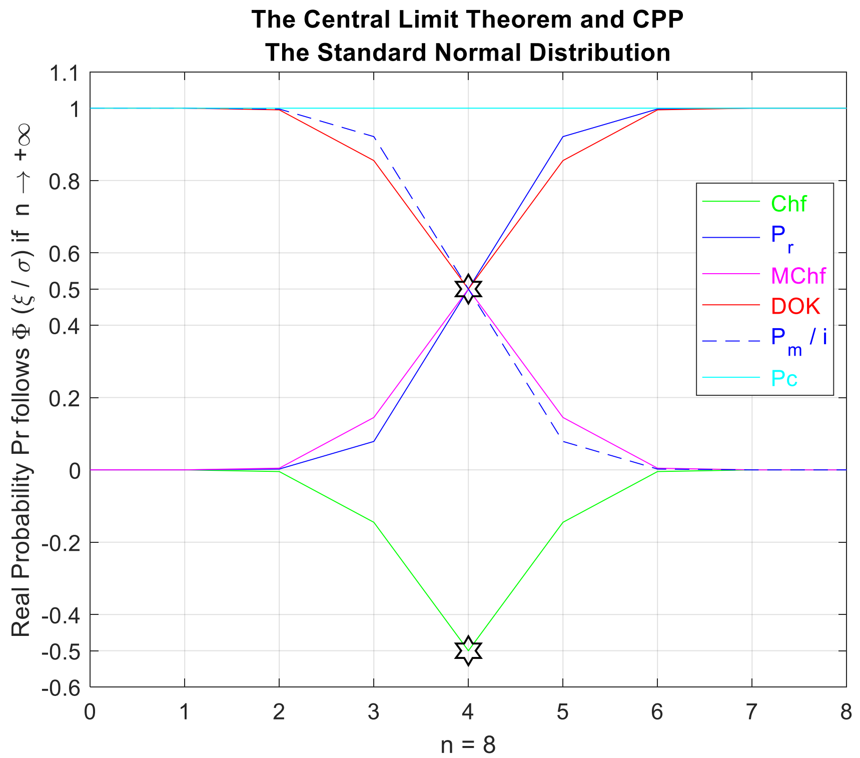

Figure 28: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution and *CPP* for a sample of size $n = 8$

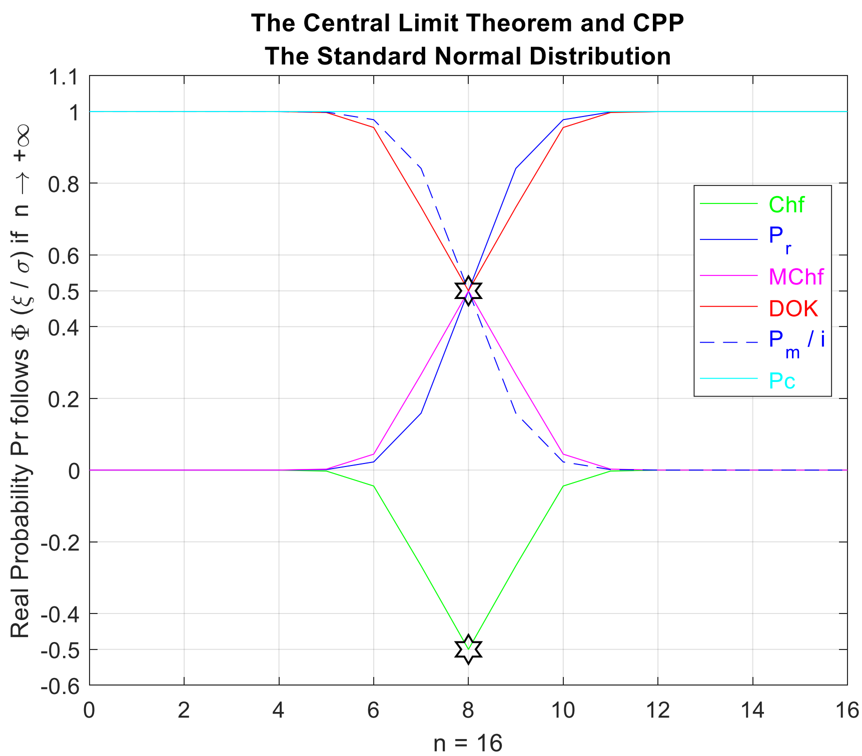

Figure 29: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution and *CPP* for a sample of size $n = 16$

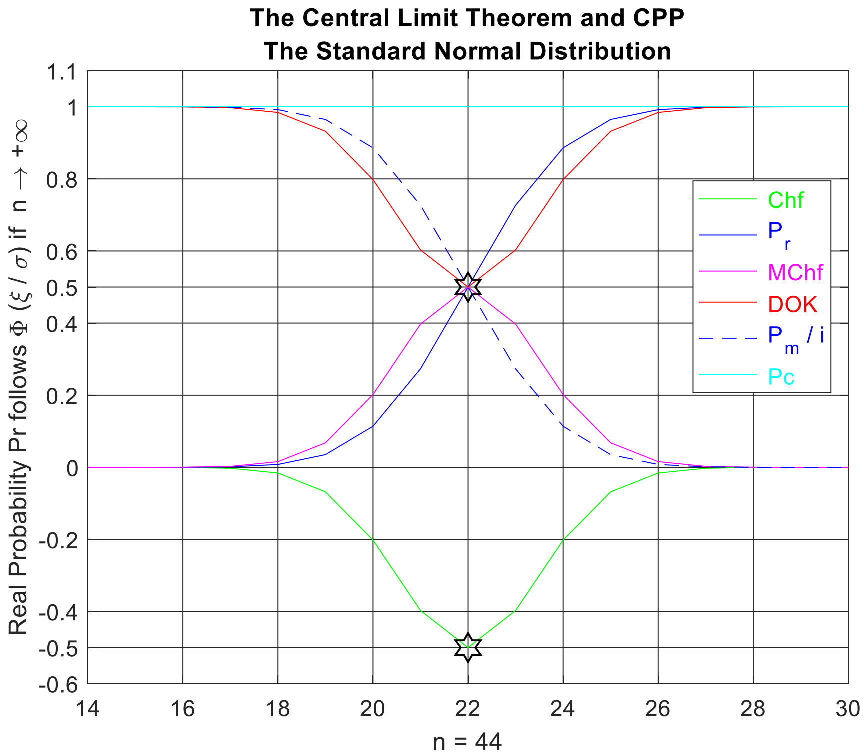

Figure 30: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution and *CPP* for a sample of size $n = 44$

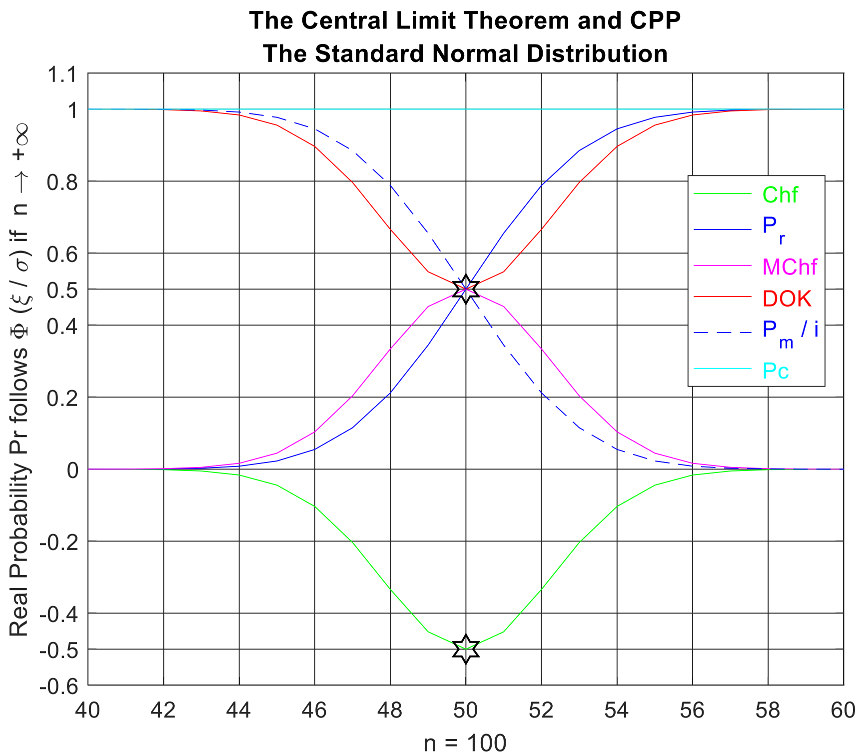

Figure 31: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution and *CPP* for a sample of size $n = 100$

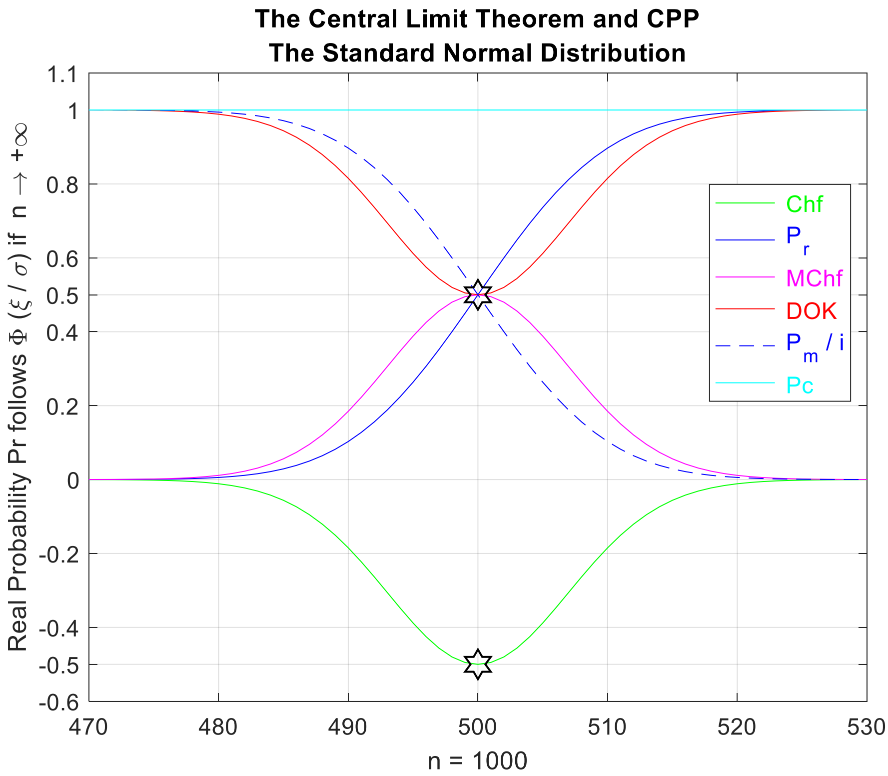

Figure 32: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution and *CPP* for a sample of size $n = 1000$

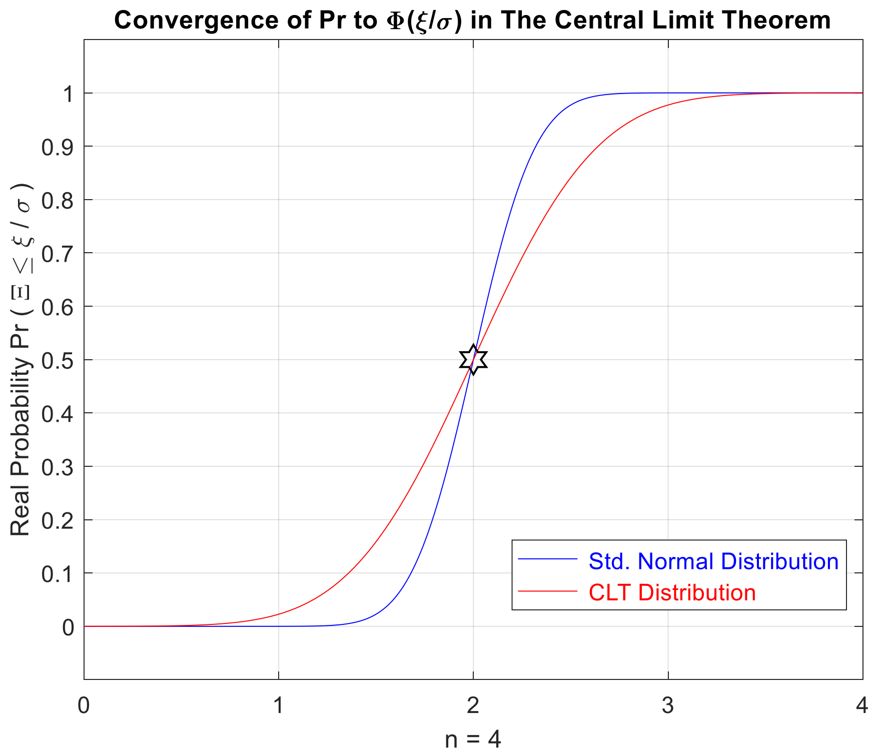

Figure 33: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution for a sample of size $n = 4$

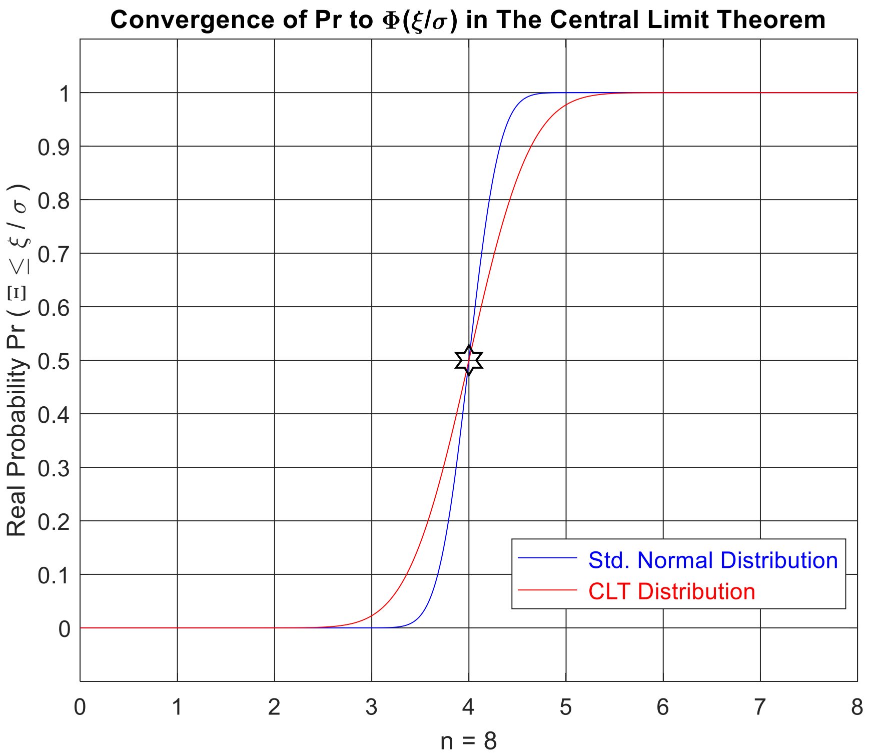

Figure 34: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution for a sample of size $n = 8$

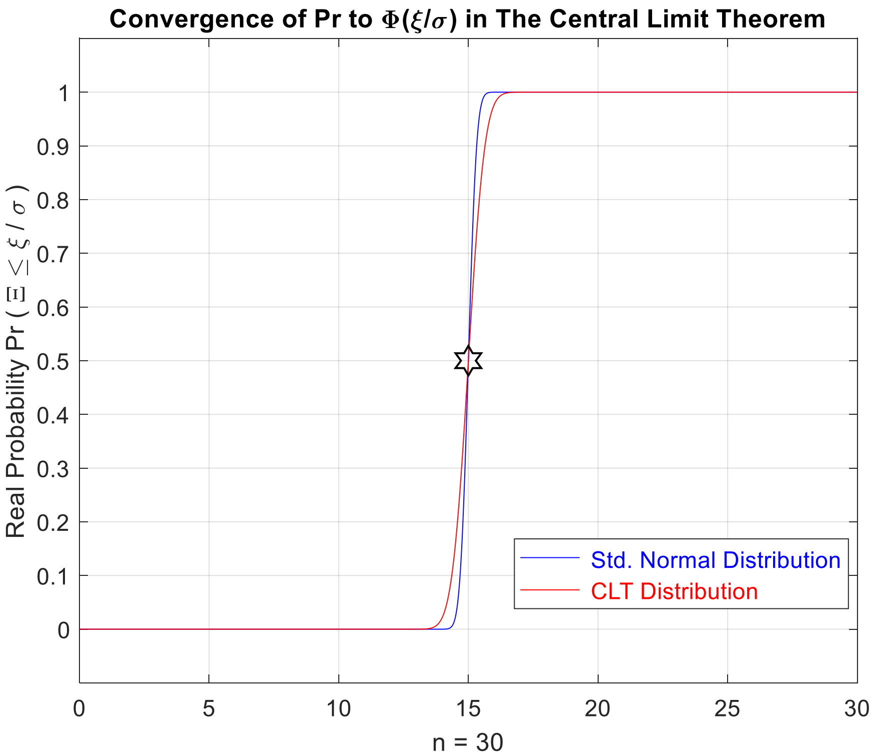

Figure 35: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution for a sample of size $n = 30$

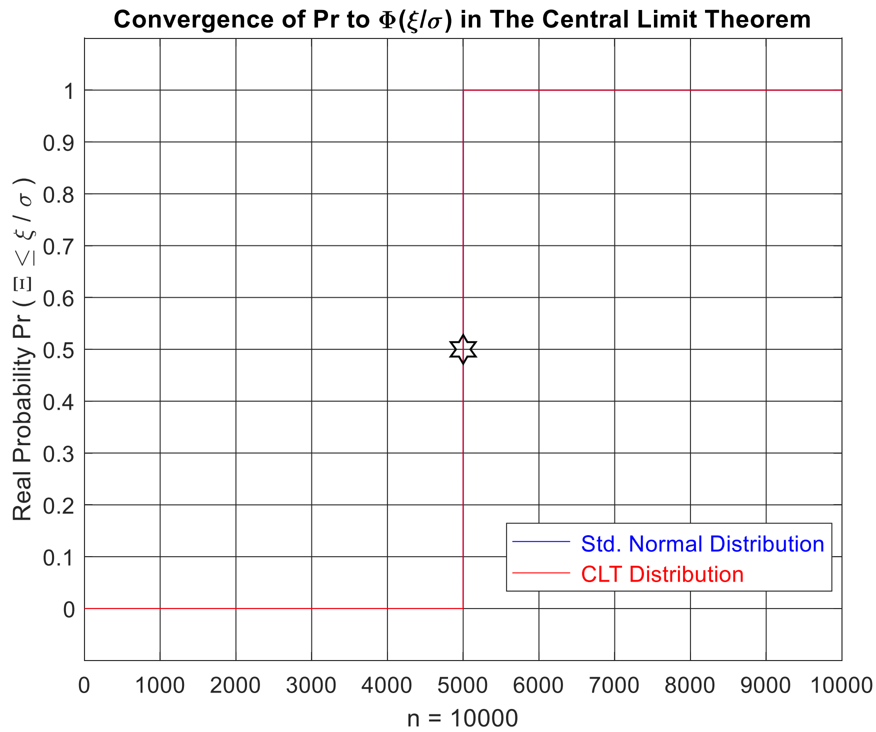

Figure 36: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution for a sample of size $n = 10000$

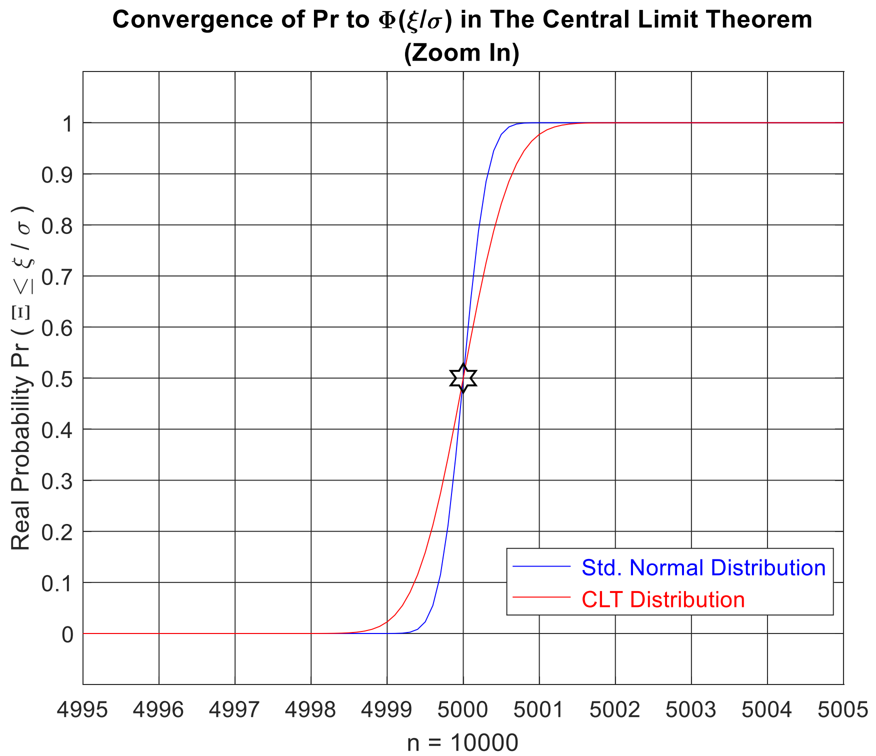

Figure 37: The increasing convergence of the probability distribution in *CLT* to the std. normal distribution for a sample of size $n = 10000$ (Zoom In)

9.3.1.1 The Simulations Interpretation

After considering here the probability distribution of the random variable $\Xi = \frac{S_n - \mu}{\sigma / \sqrt{n}}$, we can deduce a value of $P_r(\Xi)$ for each value of the random variable Ξ and for each value of the random sample size n. Figures 23 to 32 illustrate all the new prognostic model functions and prove all the mathematical derivations. We have computed and ploted for a special set of $P_r(\Xi)$ all the *CPP* parameters and components and which are: *Chf*(Ξ), *MChf*(Ξ), *DOK*(Ξ), *Pc*(Ξ), $P_m(\Xi)/i$, and showed how to calculate the correpsonding *Z*(Ξ). This is achieved with the increasing value of n by taking into consideration the cases n = 8, 16, 44, 100, and 1000 to illustrate the paradigm.

Furthermore, as it was shown and established in the original model, when $n = 0$ (before the random simulation beginning) and at n (when the simulation converges) then the degree of our knowledge (*DOK*) is 1 and the chaotic factor (*Chf* and *MChf*) is 0 since the stochastic influences and variations have either not commenced yet or they have terminated their job on the random experiment and simulation. We note from these figures that the *DOK* is maximum (*DOK* = 1) when absolute value of *Chf* which is *MChf* is minimum (*MChf* = 0), that means when the magnitude of the chaotic factor (*MChf*) decreases our certain knowledge (*DOK*) increases. Subsequently, *MChf* begins to grow during the simulation due to the intrinsic conditions thus leading to a decrease in *DOK* until they both reach 0.5 at $n/2$ in all possible cases. During the course of the nondeterministic and stochastic phenomenon ($n > 0$) we have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. The real cumulative convergence probability P_r and the real cumulative complementary divergence probability P_m/i will meet with *DOK* and *MChf* also at the point ($n/2$, 0.5) in all possible cases also. With the increase of Ξ, the *Chf* and *MChf* return to zero and the *DOK* returns to 1 where we attain the total convergence of the probability distribution of Ξ to a normal distribution as predicted by *CLT* ($P_r = 1$) as $n >> 1$ or $n \to +\infty$. At this last point, and for large n, convergence here is definite since $P_r(\Xi) = 1$ with $Pc(\Xi) = 1$ permanently, so the logical consequence of the value *DOK* = 1 follows.

We note that $n/2$ corresponds to $\Xi_{Median} = \Xi_{Mean} = \Xi_{Mode}$ of the random distribution and which are at the middle of the simulations since the binomial and normal distributions considered here are totally symmetric, therefore their corresponding graphs are perfectly symmetric.

Moreover, at each value of Ξ and n and during this entire process, we can predict with certainty all the *CPP* parameters in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ with *Pc* preserved as equal to one through a continuous compensation between *DOK* and *Chf* since $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$ in the *CPP*. This compensation is from the instant $n = 0$ (at the beginning of the random sampling and simulation) where $P_r(\Xi) = 0$ until the instant of convergence n (at the end of the random sampling and simulation) where $P_r(\Xi) = 1$. That means also that the simulation which is considered to be stochastic and random in the set $\mathcal{R}$ is now deterministic and certain in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding the contributions of $\mathcal{M}$ to the experiment happening in $\mathcal{R}$ and thus after removing and subtracting the chaotic factor from the degree of our knowledge in the equation above.

Additionally, Figures 33 to 37 show the increasing convergence probability of the random distribution to the normal (or the standard normal = $\Phi(\xi / \sigma)$) distribution with the increasing value of n by considering the values n = 4, 8, 30, and 10000, just as predicted by *CLT* that considers here the random variable Ξ.

9.3.2 The Probability of Convergence in CLT and CPP

The real convergence probability in *CLT*:

Let now

$P_r(\Xi_c) = P_{rob}(\text{Convergence in } CLT)$

$$= \frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)} = \frac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\frac{\xi}{\sigma}\right)}$$

We can write for every real number ξ and by the *CLT*:

$$\lim_{n\to+\infty} P_r(\Xi) = \lim_{n\to+\infty} P_{rob}(\Xi \le \xi/\sigma) = \lim_{n\to+\infty} P_{rob}\left[\frac{\sqrt{n}\left(S_n-\mu\right)}{\sigma} \le \frac{\xi}{\sigma}\right] = \Phi\left(\frac{\xi}{\sigma}\right)$$

where $\Phi(\xi)$ is the standard normal *CDF* evaluated at ξ.

$$\Leftrightarrow \lim_{n\to+\infty} P_r(\Xi_c) = \lim_{n\to+\infty} P_{rob}(\text{Convergence in } CLT)$$

$$= \lim_{n\to+\infty}\left\{\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right\} = \lim_{n\to+\infty}\left\{\frac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\frac{\xi}{\sigma}\right)}\right\} = 1$$

$$\Leftrightarrow \lim_{n\to+\infty} \sup_{\xi\in R}\left|P_{rob}\left[\frac{\sqrt{n}\left(S_n-\mu\right)}{\sigma} \le \frac{\xi}{\sigma}\right] - \Phi\left(\frac{\xi}{\sigma}\right)\right| = 0$$

We have $-\infty < \Xi < +\infty$ where n = 0 corresponds to the instant before the beginning of the random sampling when $P_r(\Xi_c) = P_r(\Xi) = 0$, and n corresponds to the instant at the end of the random sampling and simulation when $P_r(\Xi_c) = P_r(\Xi) = 1$.

Moreover, the value of the random difference $P_{rob}\left[\frac{\sqrt{n}\left(S_n-\mu\right)}{\sigma} \le \frac{\xi}{\sigma}\right] - \Phi\left(\frac{\xi}{\sigma}\right)$ in the simulation is null at two instances: when n = 0 (the instant before the beginning of the simulation) and at n (the instant at the end of the simulation).

The imaginary complementary divergence probability in *CLT*:

$$P_m(\Xi_c) = i\left[1 - P_r(\Xi_c)\right] = i\left[1 - \frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right] = i\left[1 - \frac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]$$

The real complementary divergence probability in *CLT*:

$$P_m(\Xi_c)/i = 1 - P_r(\Xi_c) = 1 - \frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)} = 1 - \frac{P_{rob}\left[\left(\sqrt{n}\left(S_n-\mu\right)/\sigma\right) \le \left(\xi/\sigma\right)\right]}{\Phi\left(\frac{\xi}{\sigma}\right)}$$

The complex probability and random vector in *CLT*:

$$Z(\Xi_c) = P_r(\Xi_c) + P_m(\Xi_c) = \left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right] + i\left[1 - \frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]$$

The Degree of Our Knowledge in *CLT*:

$$DOK(\Xi_c) = |Z(\Xi_c)|^2 = P_r^2(\Xi_c) + [P_m(\Xi_c)/i]^2$$

$$= \left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]^2 + \left[1 - \frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]^2$$

$$= 1 + 2iP_r(\Xi_c)P_m(\Xi_c) = 1 - 2P_r(\Xi_c)[1 - P_r(\Xi_c)] = 1 - 2P_r(\Xi_c) + 2P_r^2(\Xi_c)$$

$$= 1 - 2\left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right] + 2\left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]^2$$

$DOK(\Xi_c)$ is equal to 1 when $P_r(\Xi_c) = P_r(n=0) = 0$ and when $P_r(\Xi_c) = P_r(n) = 1$ that means at the end of the simulation.

The Chaotic Factor in *CLT*:

$$Chf(\Xi_c) = 2iP_r(\Xi_c)P_m(\Xi_c) = -2P_r(\Xi_c)[1 - P_r(\Xi_c)] = -2P_r(\Xi_c) + 2P_r^2(\Xi_c)$$

$$= -2\left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right] + 2\left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]^2$$

$Chf(\Xi_c)$ is null when $P_r(\Xi_c) = P_r(n=0) = 0$ and when $P_r(\Xi_c) = P_r(n) = 1$ that means at the end of the simulation.

The Magnitude of the Chaotic Factor *MChf* in *CLT*:

$$MChf(\Xi_c) = |Chf(\Xi_c)| = -2iP_r(\Xi_c)P_m(\Xi_c) = 2P_r(\Xi_c)[1 - P_r(\Xi_c)] = 2P_r(\Xi_c) - 2P_r^2(\Xi_c)$$

$$= 2\left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right] - 2\left[\frac{P_{rob}(\Xi \le \xi/\sigma)}{\Phi\left(\frac{\xi}{\sigma}\right)}\right]^2$$

$MChf(\Xi_c)$ is null when $P_r(\Xi_c) = P_r(n=0) = 0$ and when $P_r(\Xi_c) = P_r(n) = 1$ that means at the end of the simulation.

At any value of the random variables Ξ_c and $\Xi : -\infty < \forall \Xi < +\infty$, and for any value of the sample size *n*, the probability in *CLT* expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(\Xi_c) = [P_r(\Xi_c) + P_m(\Xi_c)/i]^2 = |Z(\Xi_c)|^2 - 2iP_r(\Xi_c)P_m(\Xi_c)$$
$$= DOK(\Xi_c) - Chf(\Xi_c)$$
$$= DOK(\Xi_c) + MChf(\Xi_c)$$
$$= 1$$

then,

$$Pc^2(\Xi_c) = [P_r(\Xi_c) + P_m(\Xi_c)/i]^2 = \{P_r(\Xi_c) + [1 - P_r(\Xi_c)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(\Xi_c) = 1 \text{ always.}$$

Hence, the prediction of the convergence probabilities of the stochastic experiments in the set $\mathcal{C}$ is permanently certain.

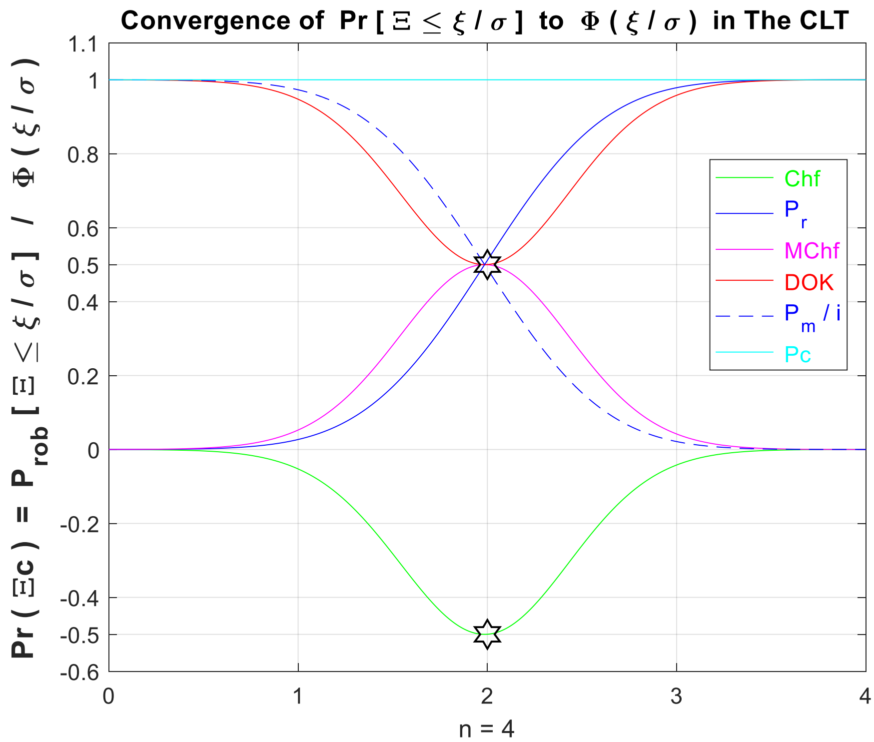

Figure 38: The increasing convergence of $P_r(\Xi_c) = P_{rob}(\Xi \le \xi/\sigma)/\Phi(\xi/\sigma)$ to 1 in *CLT* and *CPP* for a sample of size n = 4

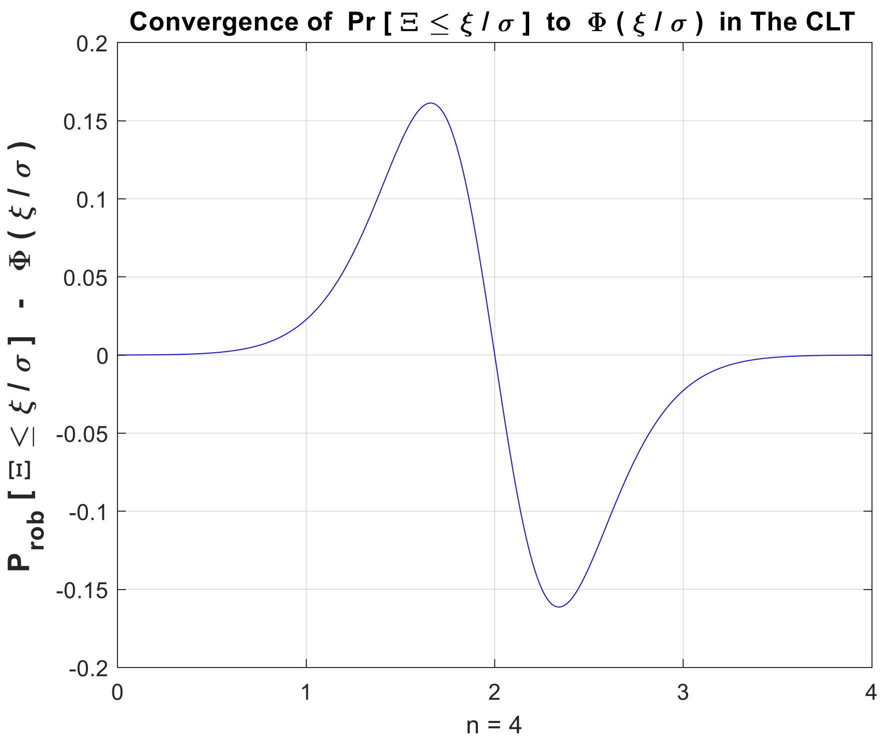

Figure 39: The increasing convergence of $P_{rob}(\Xi \leq \xi/\sigma) - \Phi(\xi/\sigma)$ to 0 in *CLT* for a sample of size $n = 4$

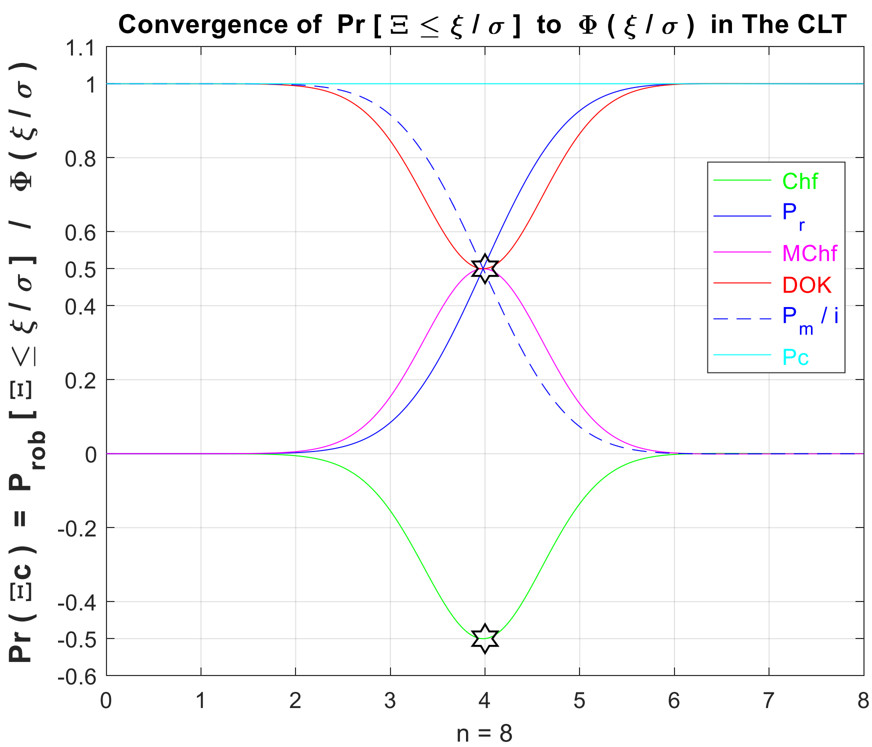

Figure 40: The increasing convergence of $P_r(\Xi_c) = P_{rob}(\Xi \leq \xi/\sigma)/\Phi(\xi/\sigma)$ to 1 in *CLT* and *CPP* for a sample of size $n = 8$

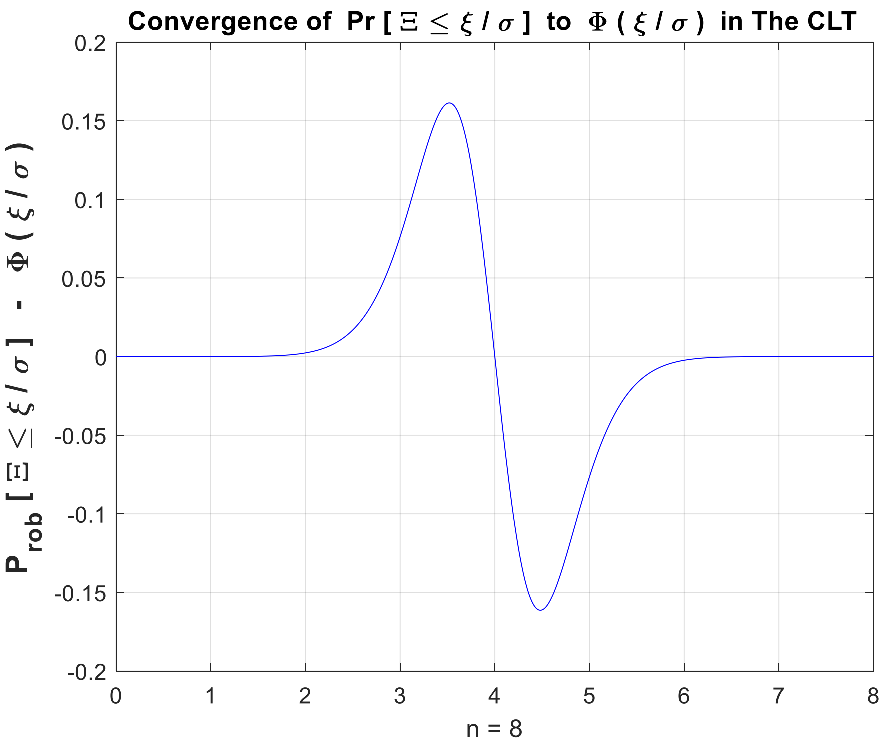

Figure 41: The increasing convergence of $P_{rob}(\Xi \le \xi/\sigma) - \Phi(\xi/\sigma)$ to 0 in *CLT* for a sample of size $n = 8$

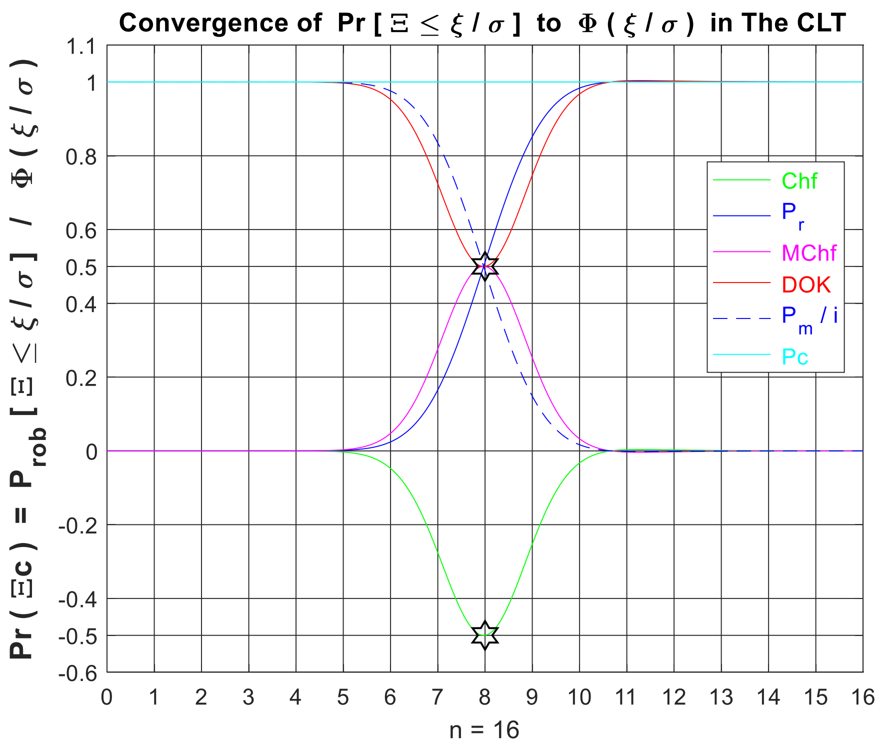

Figure 42: The increasing convergence of $P_r(\Xi_c) = P_{rob}(\Xi \le \xi/\sigma)/\Phi(\xi/\sigma)$ to 1 in *CLT* and *CPP* for a sample of size $n = 16$

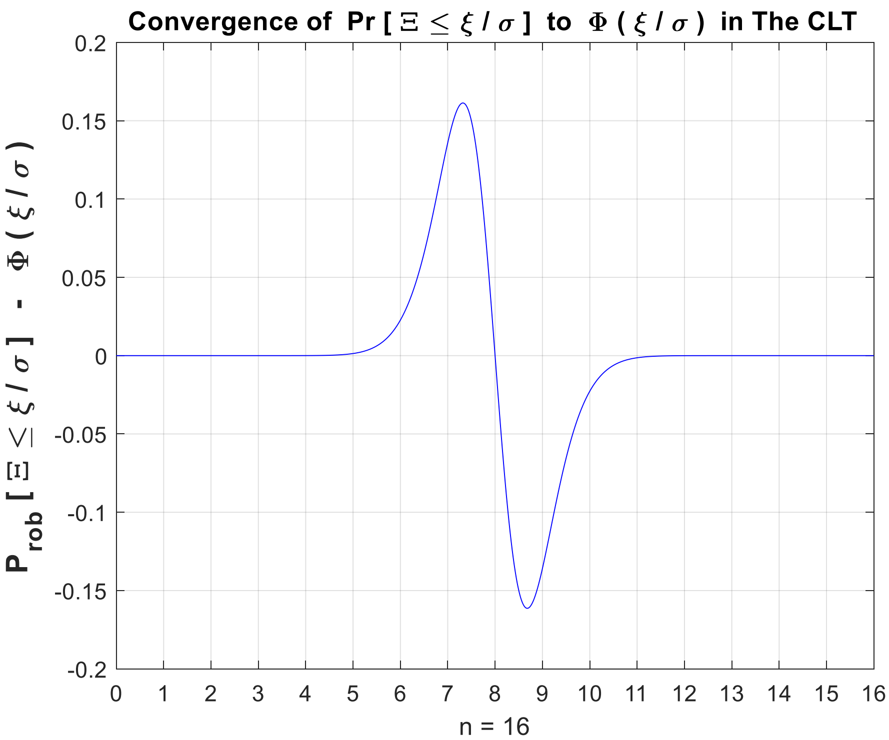

Figure 43: The increasing convergence of $P_{rob}(\Xi \le \xi/\sigma) - \Phi(\xi/\sigma)$ to 0 in *CLT* for a sample of size $n = 16$

9.3.2.1 The Simulations Interpretation

After considering at this point the random variable Ξ_c having a probability distribution of the form $P_r(\Xi_c) = P_{rob}(\Xi \le \xi/\sigma)/\Phi(\xi/\sigma)$, we can deduce a value of $P_r(\Xi_c)$ for each value of the random variables Ξ_c and Ξ : $-\infty < \forall\Xi < +\infty$ and for each value of the random sample size *n*. Figures 38, 40, and 42 illustrate all the new prognostic model functions and prove all the mathematical derivations. We have computed and drawn for a special set of $P_r(\Xi_c)$ all the *CPP* parameters and components and which are: *Chf*(Ξ_c), *MChf*(Ξ_c), *DOK*(Ξ_c), *Pc*(Ξ_c), $P_m(\Xi_c)/i$, and showed how to calculate the correpsonding *Z*(Ξ_c). This is achieved with the increasing value of *n* by taking into consideration the cases *n* = 4, 8, and 16 to illustrate the paradigm.

Furthermore, as it was demonstrated and proved in the original model, when *n* = 0 (before the random simulation beginning) and at *n* (when the simulation converges) then the degree of our knowledge (*DOK*) is 1 and the chaotic factor (*Chf* and *MChf*) is 0 since the stochastic aspects and fluctuations have either not begun yet or they have ended their task on the nondeterministic experiment and simulation. We note from these figures that the *DOK* is maximum (*DOK* = 1) when absolute value of *Chf* which is *MChf* is minimum (*MChf* = 0), that means when the magnitude of the chaotic factor (*MChf*) decreases our certain knowledge (*DOK*) increases. Subsequently, *MChf* begins to grow during the simulation due to the intrinsic conditions thus leading to a decrease in *DOK* until they both reach 0.5 at $n/2$ in all possible cases. During the course of the stochastic and random experiment ($n > 0$) we have: $0.5 \le DOK < 1$, $-0.5 \le Chf < 0$, and $0 < MChf \le 0.5$. The real cumulative convergence probability P_r and the real cumulative complementary divergence probability P_m/i will meet with *DOK* and *MChf* also at the point ($n/2$, 0.5) in all possible also. With the increase of Ξ and hence of Ξ_c, the *Chf* and *MChf* return to zero and the *DOK* returns to

1 where we attain the total convergence of $P_r(\Xi_c) = P_{rob}(\Xi \le \xi/\sigma)/\Phi(\xi/\sigma)$ distribution to one as predicted by *CLT* ($P_r = 1$) as $n >> 1$ or $n \to +\infty$. At this last point, and for large *n*, convergence here is definite since $P_r(\Xi_c) = 1$ with $Pc(\Xi_c) = 1$ permanently, so the logical consequence of the value *DOK* = 1 follows.

We note that *n*/2 corresponds to $(\Xi_c)_{Median} = (\Xi_c)_{Mean} = (\Xi_c)_{Mode}$ of the random ratio distribution and which are at the middle of the simulations since the normal distribution considered here is totally symmetric, therefore the corresponding graphs are perfectly symmetric.

Additionally, Figures 39, 41, and 43 show the increasing convergence probability of the random difference distribution $P_{rob}(\Xi \le \xi/\sigma) - \Phi(\xi/\sigma)$ to zero with the increasing value of *n* by considering the values of the sample size *n* = 4, 8, and 16, just as predicted by *CLT* for the random variable Ξ.

Moreover, at each value of Ξ_c, Ξ, and *n* and during this entire process, we can predict with certainty all the *CPP* parameters in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ with *Pc* preserved as equal to one through a continuous compensation between *DOK* and *Chf* since $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$ in the *CPP*. This compensation is from the instant *n* = 0 (at the beginning of the random sampling and simulation) where $P_r(\Xi_c) = 0$ until the instant of convergence *n* (at the end of the random sampling and simulation) where $P_r(\Xi_c) = 1$. That means also that the simulation which is considered to be stochastic and random in the set $\mathcal{R}$ is now certain and deterministic in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after taking into account the contributions of $\mathcal{M}$ to the experiment occurring in $\mathcal{R}$ and thus after eliminating and subtracting the chaotic factor from the degree of our knowledge in the equation above.

Hence and finally, what is crucial and original here, is that we have illustrated using all the simulations and graphs the convergence in *CLT* using *CPP* axioms and tools as proved in section 7.3.

X. Conclusion and Perspectives

In the current research work, the original extended Kolmogorov model of eight axioms (*EKA*) was connected and applied to the classical Central Limit Theorem. Thus, a tight link between *CLT* and the novel paradigm was executed. Consequently, the model of "Complex Probability" was more expanded beyond the scope of my fourteen earlier research studies on this subject.

Additionally, and in the novel *CPP* paradigm, the probabilities of convergence and divergence in the *CLT* procedure that correspond to each iteration cycle or sample size *n* have been determined in the three sets of probabilities which are $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by P_r, P_m, and *Pc* respectively. Accordingly, at each instance of *n*, the novel *CLT* and *CPP* parameters P_r, P_m, P_m/i, *DOK*, *Chf*, *MChf*, *Pc*, and *Z* are perfectly and surely predicted in the set of complex probabilities $\mathcal{C} = \mathcal{R} + \mathcal{M}$ with *Pc* kept as equal to 1 continuously and permanently. Also, using all these shown simulations and obtained graphs all over the entire research paper, we can visualize and quantify both the certain knowledge (expressed by *DOK* and *Pc*) and the system chaos and stochastic influences and effects (expressed by *Chf* and *MChf*) of *CLT*. Furthermore, it is important to state here that we have proved *CLT* in a novel and original way and this by using *CPP* axioms and tools. This is definitely very wonderful, fruitful, and fascinating and demonstrates once again the advantages of extending the five axioms of probability of Kolmogorov and thus the benefits and novelty of this original theory in applied mathematics and prognostics that can be called verily: "The Complex Probability Paradigm".

As a prospective and future challenges and research, we intend to more develop the novel conceived prognostic paradigm and to apply it to a diverse set of nondeterministic events like for other stochastic phenomena as in the classical theory of probability and in stochastic processes. Additionally, we will implement *CPP* more to the field of prognostic in engineering and also to the problems of random walk which have huge consequences when applied to economics, to chemistry, to physics, to pure and applied mathematics.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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