Several Rules of Prime Distribution

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ABSTRACT

In this paper probability and statistics were applied to study the distribution of prime numbers. The result not only confirmed some previously advanced conjectures, but also drew new conclusions on prime distribution.

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In this paper probability and statistics were applied to study the distribution of prime numbers. The result not only confirmed some previously advanced conjectures, but also drew new conclusions on prime distribution.

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Definition: According to the increasing order, the ith prime number is denoted as a_i; all natural numbers smaller than a_{i+1}^2 constitute the ith session, denoted as s_i; all ranges $(a_i \times j, \ a_i \times (j+1)]$ $(j \geq 0)$ on s_i is a_i long and denoted as r_j.

Theorem 1 Every composite on s_i has at least one prime factor smaller than or equal to a_i.

Proof:

Assumption there is a composite $m = b \times c$ (b, c are primes, $b>a_i$, $c>a_i$; m is on s_i).
Since $b > a_i$ and $c > a_i$,
then $b \geq a_{i+1}$, $c \geq a_{i+1}$
$b \times c \geq a_{i+1}^2$
Namely, $m \geq a_{i+1}^2$

This contradicts the definition of s_i (all numbers on s_i are smaller than a_{i+1}^2).

Therefore the above assumption does not hold. Namely, all composites on s_i have at least one prime factor smaller than or equal to a_i.

This completes the proof.

Theorem 2 If all primes on s_i cannot divide exactly a natural number m, m is either a prime or a composite greater than a_{i+1}^2.

Proof:

Assumption A natural number m on s_i) is a composite.

According to **Theorem 1**, m must have a prime factor $f \leq a_i$.

This contradicts the condition given in the theorem (all primes on s_i cannot divide exactly a natural number m). Therefore the above assumption does not hold, namely, m is either not a composite or a composite not on s_i (greater than a_{i+1}^2).

This completes the proof.

Theorem 3 Among any p consecutive natural numbers, there is one and only one number that can be divided exactly by p.

Proof:

1) *Assume* there is no number in p consecutive natural numbers that can be divided exactly by p.

Suppose the p consecutive natural numbers are $c_1, c_2, c_3 \ldots\ldots c_p$, and s is a natural number.

If $s = c_1 \bmod p$, then $p > s > 0$, $p - s > 0$, it implies that c_1 is $p - s$ less than the next multiple of p. Apparently, $p > p - s$, namely, the $(p - s)^{th}$ number after c_1 (within the scope of p consecutive natural numbers starting with c_1) there must be a multiple of p. Put in another way, there is a number in p consecutive natural numbers that can be divided exactly by p, contradicting the above assumption.

2) *Assume* there are two numbers, m and n, in p consecutive natural numbers, and m and n both can be divided exactly by p.

Suppose there are two different natural numbers x and y, $m = p \times x$, and $n = p \times y$.

$$
\begin{aligned}
m - n &= (p \times x) - (p \times y) \\
&= p \times (x - y) \\
&\geq p
\end{aligned}
$$

$m - n \geq p$ is impossible since the difference among p consecutive natural number must be smaller than p. Therefore the above assumption does not hold, namely, there are no two numbers among p consecutive natural numbers that can be divided exactly by p.

It is easy to prove, in a similar way, that there no more than two numbers among p consecutive natural numbers that can be divided exactly by p.

Therefore there is no more than one number among p consecutive natural numbers that can be divided exactly by p.

In summary, since there is at least one multiple of p among p consecutive natural numbers and the number of such multiples cannot be greater than 1, there must be one and only one number in p consecutive natural numbers that can be divided exactly by p.

This completes the proof.

Theorem 4 There are 1 to i primes on each range r_j on s_i.

Proof:

Whether $a \nmid m$ and whether $b \nmid m$ are two independent events. Probability for both $a \nmid m$ and $b \nmid m$ is the product of probability of $a \nmid m$ and probability of $b \nmid m$. Therefore the probability of m on s_i being a prime, P_m, is the product of probabilities of all primes smaller or equal to a_i cannot divide m exactly.

Since probability of $a_i \nmid m = \frac{a_i - 1}{a_i}$

$$
\begin{aligned}
P_m &= \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \frac{12}{13} \times \frac{16}{17} \times \cdots\cdots \times \frac{a_i - 1}{a_i} \\
&> \frac{1}{2} \times \frac{2}{3} \times \frac{\mathbf{3}}{\mathbf{4}} \times \frac{4}{5} \times \frac{\mathbf{5}}{\mathbf{6}} \times \frac{6}{7} \times \frac{\mathbf{7}}{\mathbf{8}} \times \frac{\mathbf{8}}{\mathbf{9}} \times \frac{\mathbf{9}}{\mathbf{10}} \times \frac{10}{11} \times \frac{\mathbf{11}}{\mathbf{12}} \times \frac{12}{13} \times \frac{\mathbf{13}}{\mathbf{14}} \times \frac{\mathbf{14}}{\mathbf{15}} \times \frac{\mathbf{15}}{\mathbf{16}} \times \frac{16}{17} \times \cdots\cdots \times \frac{a_i - 2}{a_i - 1} \times \frac{a_i - 1}{a_i} \\
&= \frac{1}{a_i} \qquad (1)
\end{aligned}
$$

Namely, $P_{\mathrm{m}} > \frac{1}{a_i}$

The prediction of the formula (1) has been confirmed by the distribution of primes (Figure 1, Table 1, 2).

Since there are a_i natural numbers on each range r_j in s_i,

Number of primes on each $r_j = P_{\mathrm{m}} \times a_i > \frac{1}{a_i} \times a_i = 1$

Namely, there is at least one prime on each range r_j in s_i.

When $j = 0$, $r_j = (0, a_i]$, obviously there are i primes on r_j in s_i.

This completes the proof of **Theorem 4**.

When $j = 1$, $r_j = (a_i, 2a_i]$, **Theorem 4** introduces **Corollary 1**.

Corollary 1 There must be a prime between a and $2a$, where a is a prime.

When $j = a_i - 1$, $r_j = (a_i^2 - a_i, a_i^2] \in s_i$. **Theorem 4** introduces **Corollary 2**.

Corollary 2 There must be at least one prime between $a^2 - a$ and a^2, where a is a prime.

When $j = a_i$, $r_j = (a_i^2, a_i^2 + a_i] \in s_i$. **Theorem 4** introduces **Corollary 3**.

Corollary 3 There must be at least one prime between a^2 and a^2+a, where a is a prime.

Since $(a_i^2, a_i^2 + a_i] \in (a_i^2, a_i^2 + 2a_i + 1] = (a_i^2, (a_i+1)^2] \in s_i$, **Corollary 3** implies that "There must be at least one prime between n^2 and $(n+1)^2$", when n is a prime. This was listed as a hard problem by Hua [1] on page 90, and now is proven. Therefore there is **Corollary 4**.

Corollary 4 There must be at least one prime between a^2 and $(a+1)^2$, where a is a prime.

Since $a_{i+1}^2 - a_i^2 = (a_{i+1} - a_i)(a_{i+1} + a_i) > 2*2a_i$ (except $a_i = 2$), the number of primes in (a_i^2, a_{i+1}^2) is $2 * 2a_i * \frac{1}{a_i} > 4$. Therefore **Theorem 4** introduces **Corollary 5**.

Corollary 5 There must be at least four primes between a_{i+1}^2 and a_i^2, where a_{i+1} and a_i are primes, $a_i \neq 2$.

According to **Theorem 4**, There are at least 1 prime on each range r_j on s_i. The differences between numbers in two adjacent ranges, r_j and r_{j+1}, are smaller than $2a_i$. Therefore **Corollary 6** is introduced.

Corollary 6 The difference between two primes on adjacent ranges in s_i is smaller than $2a_i$.

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Tabel 1: Statistics of primes on s_i and comparison with theoretical calculation. For example, in the second column, s_i ranges from 1 to 25, in which there are 8 ranges of length 3, and the upper limit is 24. According to **Formula (1)**, probability of numbers on s_i being primes is 0.333, number of primes should be 24×0.333 = 8 while the actual number is 9, and accuracy is 0.889. As s_i increases, the accuracy approaches completely accurate (1.0) (Figure 1).

s_i upper limit	9	25	49	121	169	289	361	529	841	961	1369	1681	1849	2209	2809	3481	3721	4489	5041	5329	6241	6889	7921	9409	10201
Range length a_i	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
Number of ranges	8	24	45	119	165	286	357	513	828	957	1364	1665	1845	2193	2773	3445	3717	4453	5025	5325	6205	6873	7885	9345	10185
Probablity	0.5	0.333	0.267	0.229	0.208	0.192	0.181	0.171	0.164	0.158	0.153	0.149	0.145	0.142	0.139	0.136	0.134	0.132	0.13	0.128	0.126	0.124	0.123	0.122	0.12
Calculated number of primes on s_i	4	8	12.01	27.25	34.32	54.91	64.61	87.72	135.8	151.2	208.7	248.1	267.5	311.4	385.4	468.5	498.1	587.8	653.3	681.6	781.8	852.3	969.9	1140	1222.2
Actual number of primes on s_i	4	9	14	30	38	61	71	97	144	162	218	261	282	327	403	481	518	605	674	705	807	885	997	1157	1251
Calculating accuracy	1	0.889	0.858	0.908	0.903	0.900	0.910	0.904	0.943	0.933	0.957	0.951	0.949	0.952	0.956	0.974	0.962	0.972	0.969	0.967	0.969	0.963	0.973	0.985	0.977

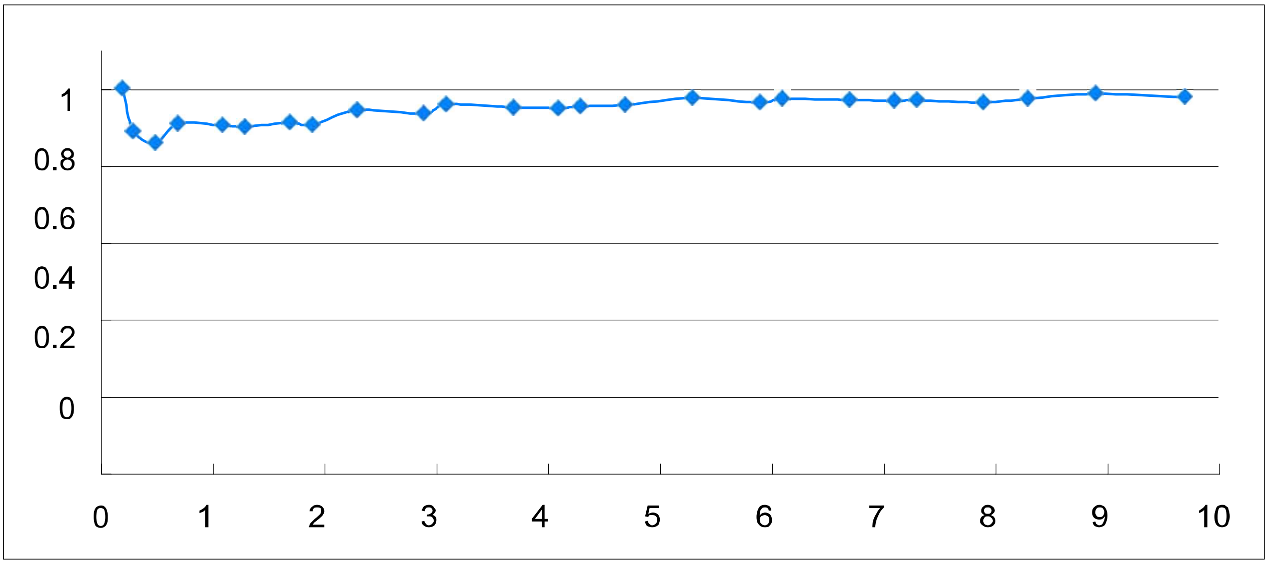

Figure 1: The accuracy of theoretical prediction approaches 1.0, as primes grow greater.

Tabel 2: Statistics of primes on different ranges on s_i. For example, in the second line, s_i ranges from 1 to 25, in which there are 8 ranges of length 3, there are 2, 1, 1, 1, 1, 1, 1, 1 primes in each range, respectively, the maximum is 2, and minimum is 1. As the session and range lengths increase, the maximum and minimum generally increase.

Upper limit of s_i	Number of ranges in s_i	Range length, a_i	Numbers of primes on each range	Max	min
9	4	2	1, 1, 1, 1,	1	1
25	8	3	2, 1, 1, 1, 1, 1, 1, 1,	2	1
49	9	5	3, 1, 2, 2, 1, 1, 1, 1, 2,	3	1
121	17	7	4, 2, 2, 1, 2, 2, 2, 1, 2, 1, 2, 2, 1, 1, 2, 2, 1,	4	1
169	15	11	5, 3, 3, 3, 2, 2, 3, 2, 2, 4, 1, 2, 2, 2, 2,	5	1
289	22	13	6, 3, 3, 3, 3, 3, 3, 3, 3, 1, 3, 2, 3, 3, 2, 2, 1, 4, 2, 2, 3, 3,	6	1
361	21	17	7, 4, 4, 4, 4, 3, 4, 2, 4, 3, 3, 4, 1, 4, 3, 4, 3, 1, 4, 2, 3,	7	1
529	27	19	8, 4, 4, 5, 3, 6, 2, 4, 3, 3, 4, 3, 4, 3, 5, 1, 4, 2, 4, 3, 3, 2, 4, 3, 4, 3, 3,	8	1
841	36	23	9, 5, 5, 5, 6, 3, 4, 5, 4, 4, 4, 4, 4, 4, 2, 5, 4, 3, 4, 4, 4, 4, 3, 2, 4, 3, 6, 3, 4, 3, 3, 3, 4, 3, 2, 5,	9	2
961	33	29	10, 6, 7, 7, 4, 6, 6, 4, 5, 6, 5, 3, 5, 5, 5, 6, 4, 4, 3, 5, 5, 4, 6, 4, 3, 5, 4, 4, 5, 4, 4, 3, 5,	10	3
1369	44	31	11, 7, 6, 6, 6, 6, 5, 6, 6, 4, 5, 5, 6, 5, 6, 4, 5, 3, 5, 7, 4, 5, 4, 5, 5, 2, 6, 5, 4, 4, 4, 5, 5, 5, 3, 6, 3, 4, 4, 6, 2, 7, 5, 1,	11	1

1681	45	37	12, 9, 8, 5, 8, 5, 8, 7, 5, 6, 6, 7, 6, 5, 4, 6, 7, 7, 5, 5, 6, 4, 5, 8, 3, 5, 6, 6, 6, 6, 3, 5, 5, 5, 6, 7, 2, 3, 6, 5, 7, 5, 5, 8, 3,	12	2
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2209	51	43	14, 9, 8, 8, 8, 8, 7, 6, 8, 6, 9, 6, 5, 8, 7, 7, 5, 8, 4, 8, 5, 6, 6, 7, 7, 7, 4, 6, 6, 6, 8, 3, 3, 9, 7, 4, 7, 8, 5, 4, 7, 5, 4, 7, 5, 3, 8, 5, 6, 7, 3,	14	3
2809	59	47	15, 9, 10, 8, 9, 9, 6, 8, 8, 9, 6, 6, 8, 8, 7, 7, 6, 7, 8, 5, 7, 8, 6, 8, 5, 6, 6, 9, 4, 5, 8, 8, 5, 7, 8, 4, 7, 6, 6, 7, 4, 5, 8, 5, 8, 7, 3, 6, 7, 6, 8, 6, 5, 3, 6, 5, 7, 9, 6,	15	3
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