Two Ways to Prove Goldbach Conjecture

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Keywords: prime number, goldbach conjecture.

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I. INTRODUCTION

The Goldbach Conjecture states that every even number greater than 2 is a sum of two primes. This conjecture has been confirmed to be true for all evens up to 6 x 10^{16} [1]. However, theoretically, this Conjecture remains unproven to this date despite tedious efforts [1; 2; 3; 4]. Here I tried to give two proofs for the Conjecture.

Proof 1:

Theorem 1. If all primes smaller than or equal to $\sqrt{a}$ cannot divide a natural number a exactly, then a is a prime.

Proof: Suppose a is a composite.

Since a is a composite, a can be expressed as a product of two natural numbers, namely, $a = b \times c$ (b and c are natural numbers).

If $b = c$, then $b = c = \sqrt{a}$. This contradicts the premise of the theorem.

If $b \neq c$, one of b and c must be smaller than $\sqrt{a}$. This contradicts the premise of the theorem, too.

Thus the above supposition (a is a composite) cannot be true, therefore a must be a prime.

This completes the proof of Theorem 1.

Denotation:

a is a natural number greater than 3.
p is an odd prime smaller than a.
P is a set of all p.
n is the number of elements in P.
p_i is the i_{th} prime.
p_t is the greatest prime that is smaller than $\sqrt{2a}$.

Theorem 2. Any natural number greater than 3 is the average of at least one pair of primes.

According to Theorem 1, if all prime factors smaller than or equal to $\sqrt{2a}$ cannot divide $2a$ exactly, $2a$ is a prime. Therefore whether $2a - p$ ($<2a$) being a prime can be sufficiently determined by dividing it using all odd primes smaller than $\sqrt{2a}$ as all values of $2a$ - p are odd. Theoretically, the minimal probability of numbers in the range (a, 2a) is calculated as

$$\begin{aligned}
\text{prob}(2a) &= \frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\frac{10}{11}\times\frac{12}{13}\times\frac{16}{17}\times\cdots\cdots\times\frac{P_t-1}{P_t} \\
&= \frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\times\frac{7}{8}\times\frac{8}{9}\times\frac{9}{10}\times\frac{10}{11}\times\frac{11}{12}\times\frac{12}{13}\times\frac{13}{14}\times\frac{14}{15}\times\frac{15}{16}\times\frac{16}{17}\times\cdots\cdots\times\frac{P_t-2}{P_t-1}\times\frac{P_t-1}{P_t} \\
&\quad \times\frac{4}{3}\times\frac{6}{5}\times\frac{8}{7}\times\frac{9}{8}\times\frac{10}{9}\times\frac{12}{11}\times\frac{14}{13}\times\frac{15}{14}\times\frac{16}{15}\times\cdots\cdots\times\frac{P_t-1}{P_t-2} \\
&= \frac{2}{P_t}\times\frac{4}{3}\times\frac{6}{5}\times\frac{8}{7}\times\frac{9}{8}\times\frac{10}{9}\times\frac{12}{11}\times\frac{14}{13}\times\frac{15}{14}\times\frac{16}{15}\times\cdots\cdots\times\frac{P_t-1}{P_t-2} \\
&= \frac{2}{P_t}\times\frac{4}{3}\times\frac{6}{5}\times\frac{10}{7}\times\frac{12}{11}\times\frac{16}{13}\times\cdots\cdots\times\frac{P_t-1}{P_{t-1}} \\
&= \frac{2}{P_t}\times\frac{5-1}{3}\times\frac{7-1}{5}\times\frac{11-1}{7}\times\frac{13-1}{11}\times\frac{17-1}{13}\times\cdots\cdots\times\frac{P_t-1}{P_{t-1}} \\
&= \frac{2}{P_t}\times\prod_{i=2}^{t-1}\frac{P_{i+1}-1}{P_i}
\end{aligned} \qquad 1)$$

As the number of different values of $2a - p$ in (a, 2a) is equal to the number of elements in P (namely, n), the minimal number of primes in form of $2a - p$ can be under-calculated as following, as prob($2a$-p) > prob($2a$) since probability of primes is increasingly reduced as the range extends and $2a > 2a - p$.

$$n\times\text{prob}(2a - p) > n\times\text{prob}(2a) = \frac{2n}{P_t}\times\prod_{i=2}^{t-1}\frac{P_{i+1}-1}{P_i} \qquad 2)$$

In Formula 2, $n = \pi(a) = O(a/\ln a)$, $P_t = O(\sqrt{2a})$, so $\frac{2n}{P_t}$ increases with a, and $\prod_{i=2}^{t-1}\frac{P_{i+1}-1}{P_i}$ monotonously increases regardless of the value of a. So it is easy to conclude that the above estimation in Formula 2 generally increases with a.

Despite such underestimations, it is clear that the estimated numbers of primes in form of ($2a$ - p) in (a, $2a$) are always greater than 1 when $a > 5$ and continue to increase, meaning that, there are at least one pair of primes, p and $2a - p$, with their average equal to a, as shown in Figure 1. This completes proof 1 of Theorem 2.

Table 1: Although it may be as small as a fraction of 1 in the second example, the estimated numbers of primes in form of $2a - p$ in (a, 2a) are always greater than 1 in other examples, indicating that there are at least one pair of primes, p and $2a$ - p, with their average equal to every natural number a.

a	$2a$	$\sqrt{2a}$	n	p_t	Minimal calculated probability of being primes of numbers in form ($2a$ - p)	estimated number of primes in form of ($2a$ - p)	actual number of primes in (a, $2a$)	annotations
4	8	2.8	1	2	1	1	2	Underestimated
5	10	3.2	1	3	0.89	0.89	1	Underestimated
6	12	3.5	2	3	0.89	1.78	2	Underestimated

7	14	3.7	2	3	0.89	1.78	2	Underestimated
8	16	4	3	3	0.89	2.67	2	***Overestimated***
9	18	4.2	3	3	0.89	2.67	3	Underestimated
10	20	4.5	3	3	0.89	2.67	4	Underestimated
11	22	4.7	3	3	0.89	2.67	3	Underestimated
12	24	4.9	4	3	0.89	3.56	4	Underestimated
13	26	5.1	4	5	0.64	2.56	3	Underestimated
14	28	5.3	5	5	0.64	3.2	3	***Overestimated***
15	30	5.5	5	5	0.64	3.2	4	Underestimated
16	32	5.7	5	5	0.64	3.2	5	Underestimated
17	34	5.8	5	5	0.64	3.2	4	Underestimated
18	36	6	6	5	0.64	3.84	4	Underestimated
19	38	6.2	6	5	0.64	3.84	4	Underestimated
20	40	6.3	7	5	0.64	4.48	4	***Overestimated***
21	42	6.5	7	5	0.64	4.48	5	Underestimated
22	44	6.6	7	5	0.64	4.48	6	Underestimated
23	46	6.8	7	5	0.64	4.48	5	Underestimated
24	48	6.9	8	5	0.64	5.12	6	Underestimated
25	50	7.1	8	7	0.52	4.16	6	Underestimated
50	100	10	14	7	0.52	7.28	10	Underestimated
100	200	14.1	24	13	0.472	11.3	21	Underestimated
200	400	20	45	19	0.396	17.8	32	Underestimated
300	600	24.5	61	23	0.398	36.2	47	Underestimated
400	800	28.3	77	23	0.398	30.7	61	Underestimated
500	1000	31.6	94	31	0.355	33.4	73	Underestimated
600	1200	34.6	108	31	0.355	38.3	87	Underestimated
700	1400	37.4	124	37	0.322	39.9	97	Underestimated
800	1600	40	138	37	0.322	44.4	112	Underestimated
900	1800	42.4	153	41	0.297	45.4	124	Underestimated
1000	2000	44.7	167	43	0.303	50.6	135	Underestimated
2000	4000	63.2	302	61	0.267	80.6	247	Underestimated
3000	6000	77.5	429	73	0.257	110.3	353	Underestimated
4000	8000	89.4	549	89	0.25	137.3	457	Underestimated
5000	10000	100	668	97	0.236	157.7	560	Underestimated
6000	12000	109.5	782	109	0.227	177.5	655	Underestimated
7000	14000	118.3	899	113	0.244	219.4	752	Underestimated
8000	16000	126.5	1006	113	0.244	245.5	855	Underestimated
9000	18000	134.2	1116	131	0.227	253.3	947	Underestimated
10000	20000	141.4	1228	139	0.229	281.2	1033	Underestimated
100000	200000	447.2	9591	443	0.152	1457.8	8392	Underestimated
1000000	2000000	1414.2	78497	1409	0.1283	10071.1	70435	Underestimated
10000000	20000000	4472.1	664578	4463	0.1100655	73147.1	606028	Underestimated
100000000	200000000	14142.1	5761454	14107	0.09677885	557586.9	5317481	Underestimated

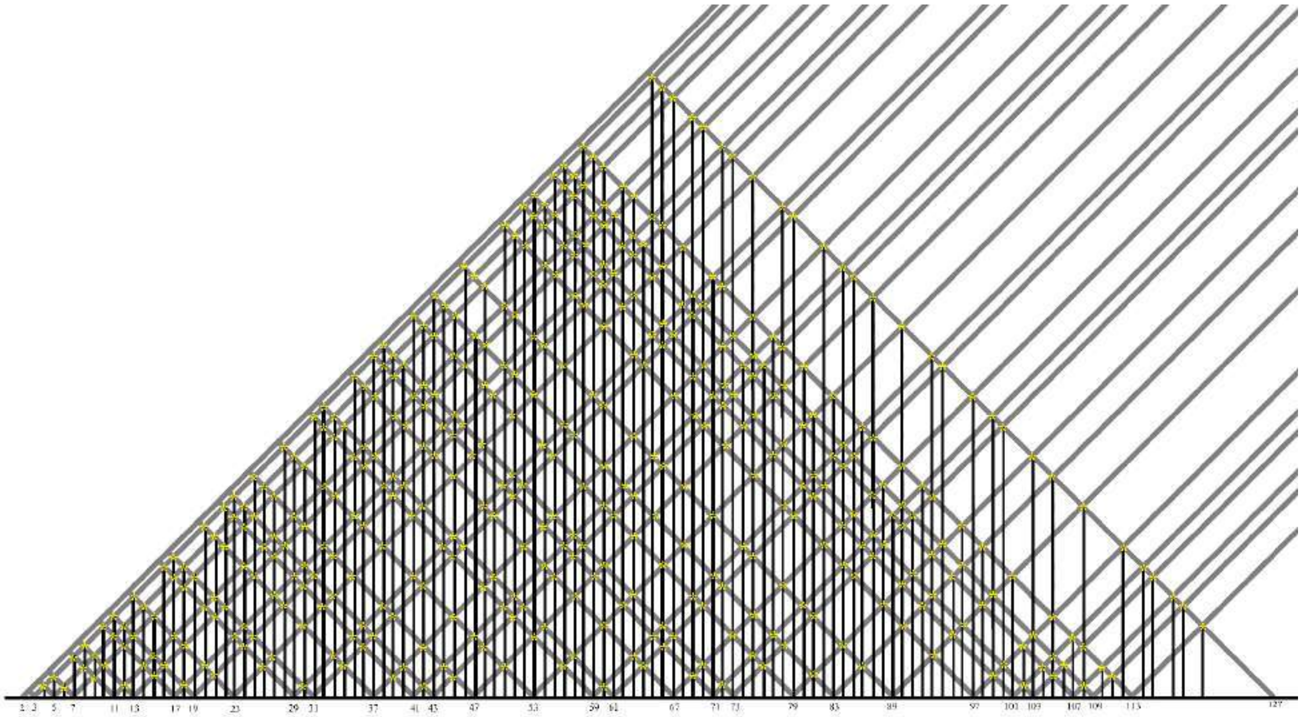

Figure 1: Every asterisk is the apex of an isosceles triangle, the left and right base angles are two primes, while base's midpoint is a natural number that is the averages of the two primes. The above rules applies for every natural number ≥4.

Proof 2:

Case 1. When $X_n = 30$

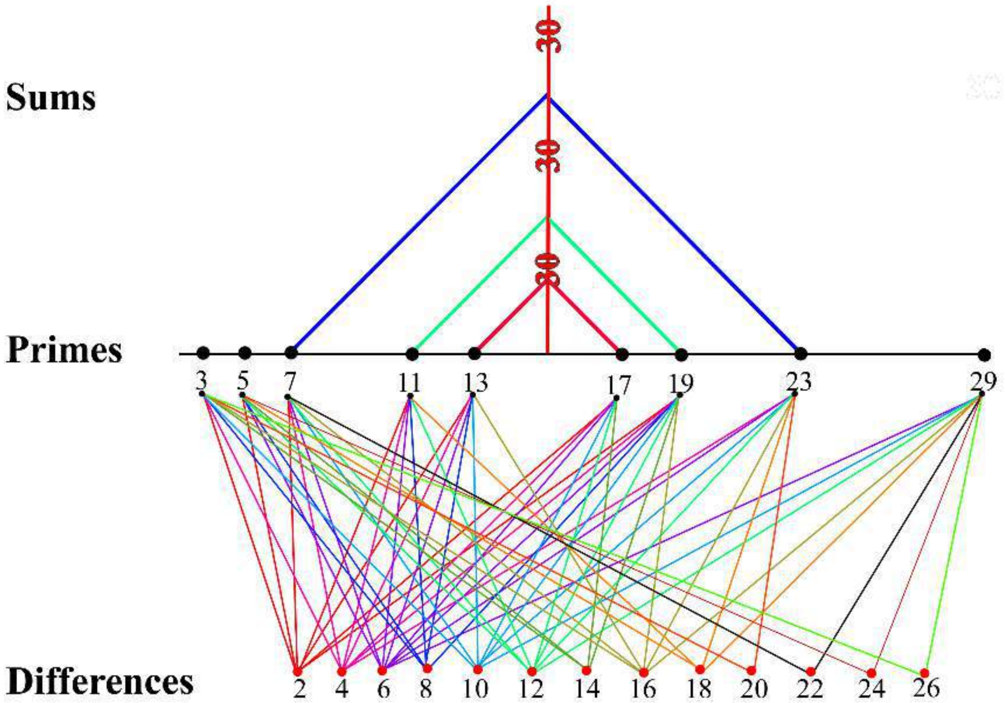

Figure 2: In the middle, odd primes smaller than 30 are plotted on an axis. Above the axis, six primes are paired and have sums equal to 30. Below the axis, even differences ranging from 2 to 26 are connected to their related primes.

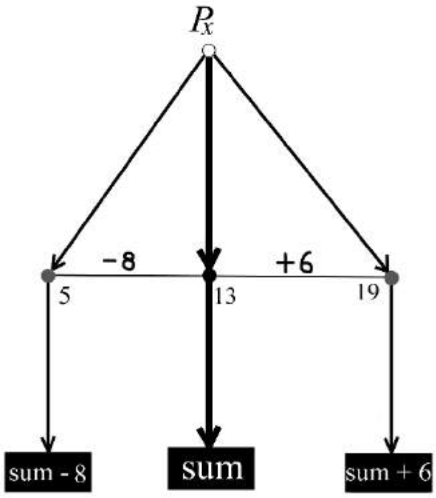

Figure 3: The sum of two primes, P_x and 13, may be decreased or increased by the difference between two primes through replacing the latter number (13) with another prime smaller or greater than it. If putting P_x and 13 together is termed as pairing, then replacing 13 in the pair with a different prime can be termed as re-pairing. Such re-pairing generates a sum different than the original one.

As proven previously [5], a super product of a prime tends to collect more prime pairs with sums equal to itself. X_4 = 2 x 3 x 5 = 30 [6], is equal to the sums of three different pairs of primes (Figure 2). For example, 7 and 23 constitute a prime pair that has a sum of 30. 23, a half of the pair, is connected with seven different even differences, including 4, 6, 10, 12, 16, 18, and 20 in Figure 2. Such connections imply that seven different even sums including 26, 24, 20, 18, 14, 12 and 10 can be obtained by replacing 23 with primes including 19, 17, 13, 11, 7, 5, and 3, respectively, while the other half of the pair, 7, remains static. In the meantime, 7, the other half of the pair, is connected with six different even differences, including 2, 4, 6, 10, 12, and 22. Such connections imply that six different even sums including 28, 34, 36, 40, 42, and 52 can be obtained by replacing 7 with primes including 5, 11, 13, 17, 19, and 29, respectively, while the other half of the pair, 23, remains static.

Table 1: The differences of primes 7 and 11 from greater primes in Figure 2 constitute a consecutive even array ranging from 2 to 12.

Prime	Differences	Annotation
7	4, 6, 10, 12, 16, 22	interrupted array
11	2, 6, 8, 12, 18	interrupted array
Summary of all differences	2, 4, 6, 8, 10, 12	consecutive array

The existence of such a consecutive even difference array ranging from 2 to 12 implies that every evens in [30, 42] can be expressed as sums of two primes in Figure 2, after re-pairing demonstrated in Figure 3.

Table 2: The differences of primes 23 and 19 from smaller primes in Figure 2 constitute a consecutive even array ranging from 2 to 20.

Prime	Differences	Annotation
23	4, 6, 10, 12, 16, 18, 20	interrupted array
19	2, 6, 8, 10, 12, 14	interrupted array
Summary	2, 4, 6, 8, 10, 12, 14, 16, 18, 20	consecutive array

The existence of such a consecutive array of even differences ranging from 2 to 20 implies that all evens in [10, 30] can be expressed as sums of two primes in Figure 2, through re-pairing demonstrated in Figure 3.

In summary, the above two observations prove that all evens in [10, 42] can be expressed as sums of two primes in Figure 2.

Case 2. When X_n = 210

X_5 = 2 x 3 x 5 x 7 = 210, is equal to the sums of 19 different pairs of primes (Figure 4). For example, 11 and 199 constitute a prime pair that has a sum of 210. 199, a half of the pair, is connected with 41 different even differences ranging from 2 to 196 (the list is omitted for simplicity) (Table 3). Such connections imply that 41 different even sums ranging from 14 to 210 can be obtained by replacing 199 with other smaller primes in Figure 4, through re-pairing demonstrated in Figure 3. 11, the other half of the pair, is connected with 41 different even differences ranging from 2 to 188 (Table 4). Such connections imply again that, by replacing 11 with greater primes in Figure 4, 41 different even sums ranging from 212 to 398 can be obtained (the list is omitted for simplicity).

Analysis of the above cases suggests that an array of consecutive evens differences in certain range guarantees all evens within certain range centered on super product of a prime to be expressed as sums of at least one pair of primes through re-pairing demonstrated in Figure 3.

Table 3: Differences of seven primes from smaller primes in Figure 4. Differences of these primes from other primes in Figure 4 constitute a consecutive array of evens ranging from 2 to 196.

Prime	Differences	
199	2, 6, 8, 18, 20, 26, 32, 36, 42, 48, 50, 60, 62, 68, 72, 86, 90, 92, 96, 98, 102, 110, 116, 120, 126, 128, 132, 138, 140, 146, 152, 156, 158, 162, 168, 170, 176, 180, 182, 186, 188, 192, 194, 196	interrupted array
197	4, 6, 16, 18, 24, 30, 34, 40, 46, 48, 58, 60, 66, 70, 84, 88, 90, 94, 96, 100, 108, 114, 118, 124, 126, 130, 136, 138, 144, 150, 154, 156, 160, 166, 168, 174, 178, 180, 184, 186, 190, 192, 194	interrupted array
193	2, 12, 14, 20, 26, 30, 36, 42, 44, 54, 56, 62, 66, 80, 84, 86, 90, 92, 96, 104, 110, 114, 120, 122, 126, 132, 134, 140, 146, 150, 152, 156, 162, 164, 170, 174, 176, 180, 182, 186, 188, 190	interrupted array
191	10, 12, 18, 24, 28, 34, 40, 42, 52, 54, 60, 64, 78, 82, 84, 88, 90, 94, 102, 108, 112, 118, 120, 124, 130, 132, 138, 144, 148, 150, 154, 160, 162, 168, 172, 174, 178, 180, 184, 186, 188	interrupted array
181	2, 8, 14, 18, 24, 30, 32, 42, 44, 50, 54, 68, 72, 74, 78, 80, 84, 92, 98, 102, 108, 110, 114, 120, 122, 128, 134, 138, 140, 144, 150, 152, 158, 162, 164, 168, 170, 174, 176, 178	interrupted array
179	6, 12, 16, 22, 28, 30, 40, 42, 48, 52, 66, 70, 72, 76, 78, 82, 90, 96, 100, 106, 108, 112, 118, 120, 126, 132, 136, 138, 142, 148, 150, 156, 160, 162, 166, 168, 172, 174, 176	interrupted array
151	2, 12, 14, 20, 24, 38, 42, 44, 48, 50, 54, 62, 68, 72, 78, 80, 84, 90, 92, 98, 104, 108, 110, 114, 120, 122, 128, 132, 134, 138, 140, 144, 146, 148	interrupted array
Summary of the above differences	2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128, 130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158, 160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188, 190, 192, 194, 196	consecutive even differences

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The existence of such a consecutive even difference array ranging from 2 to 196 implies that all evens in [14, 210] can be expressed as sums of two primes in Figure 4.

Table 4: Differences of seven primes from greater primes in Figure 4. All the differences of these primes from other primes in Figure 4 constitute a consecutive array of evens ranging from 2 to 188.

Prime	Differences	Annotation
11	2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60, 62, 68, 72, 78, 86, 90, 92, 96, 98, 102, 116, 120, 126, 128, 138, 140, 146, 152, 156, 162, 168, 170, 180, 182, 186, 188	interrupted array
13	4, 6, 10, 16, 18, 24, 28, 30, 34, 40, 46, 48, 54, 58, 60, 66, 70, 76, 84, 88, 90, 94, 96, 100, 114, 118, 124, 126, 136, 138, 144, 150, 154, 160, 166, 168, 178, 180, 184, 186	interrupted array
17	2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56, 62, 66, 72, 80, 84, 86, 90, 92, 96, 110, 114, 120, 122, 132, 134, 140, 146, 150, 156, 162, 164, 174, 176, 180, 182	interrupted array
19	4, 10, 12, 18, 22, 24, 28, 34, 40, 42, 48, 52, 54, 60, 64, 70, 78, 82, 84, 86, 90, 94, 108, 112, 118, 120, 130, 132, 138, 144, 148, 154, 160, 162, 172, 174, 178, 180	interrupted array
23	6, 8, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 60, 66, 74, 78, 80, 82, 86, 90, 104, 108, 114, 116, 126, 128, 134, 140, 144, 150, 156, 158, 168, 170, 174, 176	interrupted array
29	2, 8, 12, 14, 18, 24, 30, 32, 36, 42, 44, 50, 54, 60, 68, 72, 74, 76, 80, 84, 98, 102, 108, 110, 120, 122, 128, 134, 138, 144, 150, 152, 162, 164, 168, 170	interrupted array
31	6, 10, 12, 16, 22, 28, 30, 34, 40, 42, 48, 52, 58, 66, 70, 72, 74, 78, 82, 96, 100, 106, 108, 118, 120, 126, 132, 136, 142, 148, 150, 160, 162, 166, 168	interrupted array
Summary of all above differences	2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96,98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128, 130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158, 160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188	**consecutive** array

The existence of such a consecutive even difference array ranging from 2 to 188 implies that all evens in [210, 398] can be expressed as sums of two primes in the Figure 4.

In summary, all evens in [14, 398] can be expressed as sums of two primes in Figure 4.

II. CASE SUMMARY

In the above analyses, existence of a consecutive array of even differences makes it possible for any evens within certain range centered on a super product of a prime to be expressed as sums of two primes. The premise for such a generalization is that as long as all even differences are related to at least one of paired primes. This guarantees all evens within a certain range centered on a super product of a prime to be expressed as sums of two primes, after repairing demonstrated in Figure 3. Such premise appears to be no challenge at all, as 2 primes are enough to generate such an array in case X_4 = 30 (Table 1-2) while 7 primes are enough to generate such an array in case of X_5 = 210 (Table 3-4). As super product increases expotentially, the number of prime pairs increases correspondingly, this makes re-pairing and generating more evens as sums of two primes easier. It is intriguing that the value of super product of primes increase into the infinite as primes do while the differences among primes are conservative and can be passed into the infinite, as shown in Figures 4-5 in [6]. It seems hard to find an upper limit for the applicable scope of the above generalization.

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This completes proof 2 of Theorem 2.

A corollary can be inferred from Theorem 2. Namely,

Corollary 1: Every even number greater than 4 is equal to the sum of two primes.

Corollary 1 is an equivalent of Goldbach Conjecture.

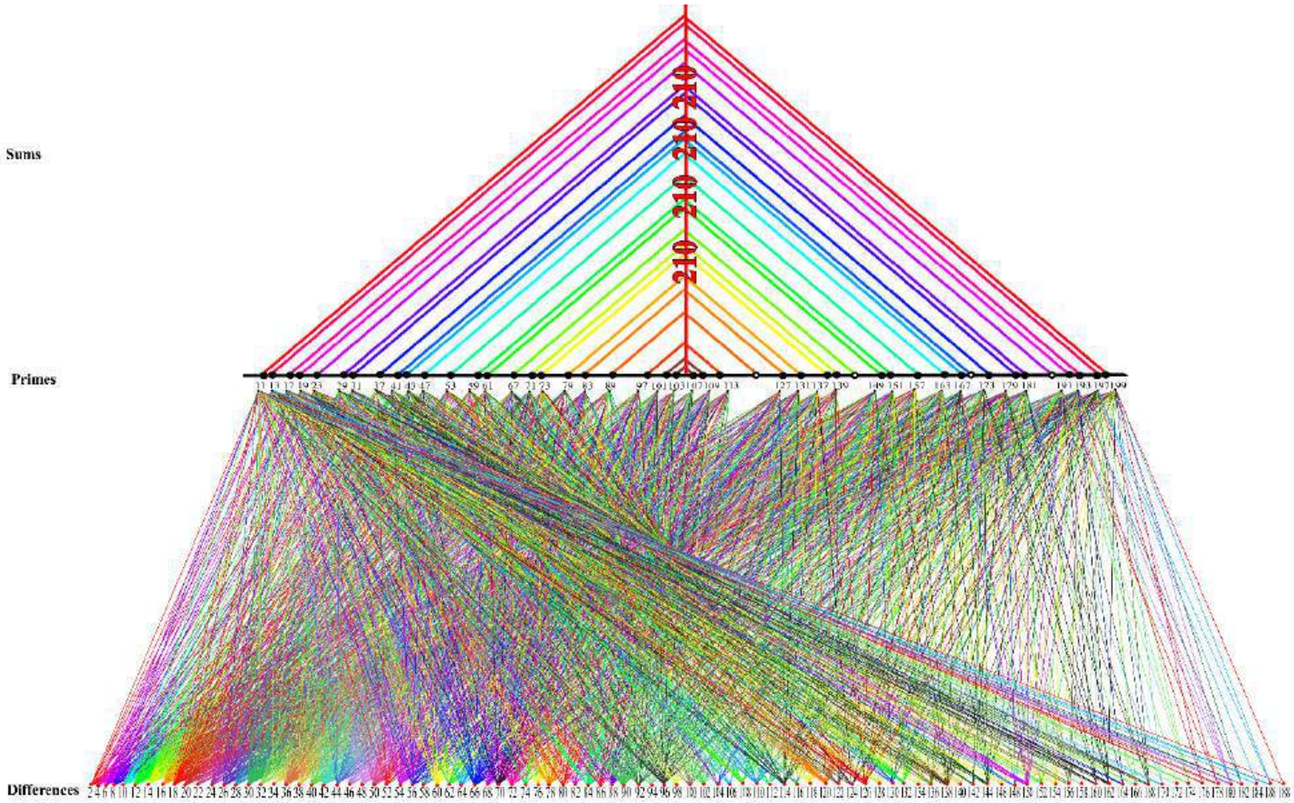

Figure 4: In the middle, odd primes smaller than 210 are plotted on an axis. Above the axis, 38 primes are paired and have sums equal to 210. Below the axis, even differences ranging from 2 to 188 are connected to their related primes.

III. CONCLUSIONS

The Goldbach Conjecture is proven true. This will help to accelerate related research on primes.

CONFLICTS OF INTERESTS

The author declares no conflicts of interests regarding the publication of this paper.

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