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ABSTRACT

In 2013, Z. Cheng and H. Gao introduced the writhe polynomial of virtual knots by using coloring of Gauss diagrams and L. H. Kauffman introduced the affine index polynomial of virtual knots by using the Cheng coloring. In 2016, I introduced the zero polynomial of virtual knots, and in 2017, Z. Cheng generalized the writhe polynomial, the affine index polynomial and the zero polynomial. These invariants are obtained by deffining the weight of a crossing for a virtual knot diagram suitably. We give a grading of virtual knots by using the weights of crossings. This reveals a relationship between virtual knots and we can get grade tree diagrams of virtual knots. It enables us to compare virtual knots with many crossings more easily.

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A Grading Virtual Knots and its Applications

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ABSTRACT

In 2013, Z. Cheng and H. Gao introduced the writhe polynomial of virtual knots by using coloring of Gauss diagrams and L. H. Kauffman introduced the affine index polynomial of virtual knots by using the Cheng coloring. In 2016, I introduced the zero polynomial of virtual knots, and in 2017, Z. Cheng generalized the writhe polynomial, the affine index polynomial and the zero polynomial. These invariants are obtained by deffining the weight of a crossing for a virtual knot diagram suitably. We give a grading of virtual knots by using the weights of crossings. This reveals a relationship between virtual knots and we can get grade tree diagrams of virtual knots. It enables us to compare virtual knots with many crossings more easily.

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I. INTRODUCTION

We will grade virtual knots by using degrees of crossings in a virtual knot. It gives an invariant of virtual knots and grade tree diagrams of virtual knots which show relations between virtual knots. In this paper all knots and virtual knots are assumed to be oriented if it is not stated.

A. Henrich ([5]) defined a sequence of virtual knot invariants which are Vassiliev invariants of degree 1. In particular a polynomial invariant of virtual knots was obtained by taking the sum of terms after assigning a weight to each crossing.

Based on the Manturov's parity axioms ([10]), Z. Cheng ([1]) introduced an odd writhe polynomial $f_K(t)$ of virtual knots K, which is useful to distinguish some virtual knot from its inverse and mirror image. Z. Cheng considered the Gauss diagram of a virtual knot and used odd chords and coloring of arcs to define the polynomial. Z. Cheng and H. Gao ([3]) generalized the odd writhe polynomial to a polynomial $W_K(t)$.
L. H. Kauffman introduced the affine index polynomial of a virtual knot based on the Cheng coloring of a virtual knot diagram ([9]). An affine index polynomial $P_K(t)$ is given in the form

$$P_K(t) = \sum_c s(c)(t^{w_K(c)} - 1),$$

where $s(c)$ and $w_K(c)$ are the sign and the weight of a crossing c respectively. $W_K(t)$ and $P_K(t)$ differ slightly so that $W_K(t) = (P_K(t) + Q_K)t$, where Q_K is a numerical invariant of virtual knots arising from writhe ([3]).

The affine index polynomial cannot distinguish virtual knots if they are related by Δ-moves. If two virtual knots are related by a sequence of Δ-moves then they have the same value for Vassiliev invariants of degree less than 2 ([6]). Since the affine index polynomial gives us a sequence of Vassiliev invariants of degree 1, we may not distinguish two virtual knots which are related by Δ-moves. I introduced the zero polynomial to distinguish virtual knots which are not distinguished with the affine index polynomial([7]) and Cheng introduced a two variable function, associated to a virtual knot, which generalize the Henrich's polynomial, the odd writhe polynomial, the affine index polynomial and the zero polynomial ([2]).

L. H. Kauffman introduced virtual knots and showed many invariants of knots can be naturally extended for virtual knots. A virtual knot is a generalization of a knot and it is motivated from a knot in a thickened surface and from realizing Gauss code which needs virtual crossings ([8]). A *virtual knot diagram* is a knot diagram allowed to have virtual crossings, which are denoted by singular points surrounded by small circles as shown in Figure 1. The virtual knot diagram in Figure 1 has three crossings and one virtual crossing.

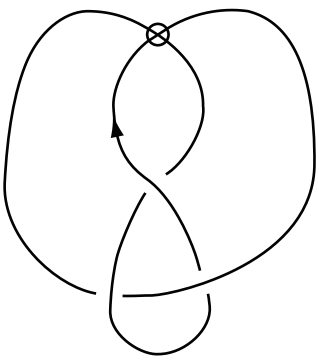

Figure 1.

We define the *sign* of a crossing of a virtual knot diagram as shown in Figure 2. A crossing is said to be *positive* (*negative*) if its sign is $+$ ($-$). We denote the sign of a crossing c by $s(c)$. The *writhe* $w(K)$ of a virtual knot diagram K is defined to be the sum of signs of all crossings of K.

Figure 2: The sign of a crossing

The *Gauss diagram* of a virtual knot diagram K is defined to be an oriented circle with chords corresponding to crossings of K. The two endpoints of a chord correspond

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to the preimages of the crossing of K. A chord corresponding to a crossing c is oriented from the preimage of the over crossing point of c to the preimage of the under crossing point of c. A chord is assumed to have the sign of the crossing corresponding to the chord. See Figure 3, which illustrates a virtual knot with two negative crossings c_1, c_2 and with a positive crossing c_3 and its Gauss diagram with three chords corresponding to the three crossings. We denote the Gauss diagram of K by $G(K)$. A virtual knot can be represented as a Gauss diagram and vice versa ([4, 8]).

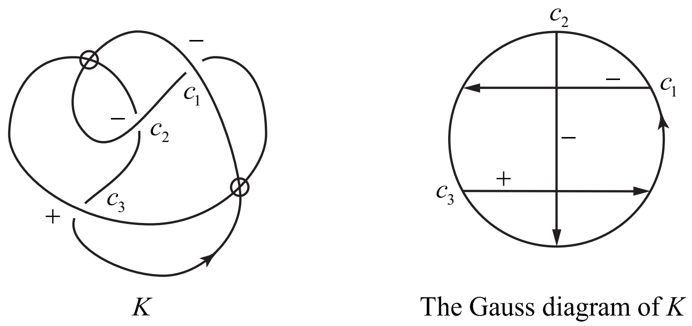

Figure 3

The moves of diagrams in Figure 4 and Figure 5 are called *Reidemeister moves* and *virtual moves* respectively.

A sequence of Reidemeister moves and virtual moves is called a *virtual isotopy*. A *virtual knot* is defined to be the virtual isotopy class of a virtual knot diagram. If there is a virtual isotopy between two virtual knot diagrams then they are said to be *equivalent* or *virtually isotopic*.

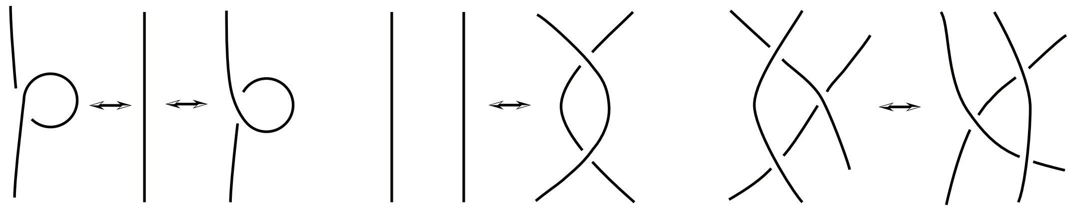

Figure 4: Reidemeister moves

A Gauss diagram G can be represented as virtual knot diagrams in many ways. All these virtual knot diagrams are virtually isotopic ([4]). In Gauss diagrams, the Reidemeister moves can be represented as a sequence of moves shown in Figure 6 ([4, 11]). From now on, ϵ in a Gauss diagram will denote the sign of a chord which can be either $+$ or $-$.

For each given virtual knot, we can get a grade tree diagram, which is invariant of virtual knots. In the grade tree diagram, usually we can get many simpler virtual knots obtained from the given one. If two virtual knots have many crossings then it can take much time to verify whether they are virtually isotopic or not. If we draw the grade tree diagrams of the two virtual knots, we can compare them more easily.

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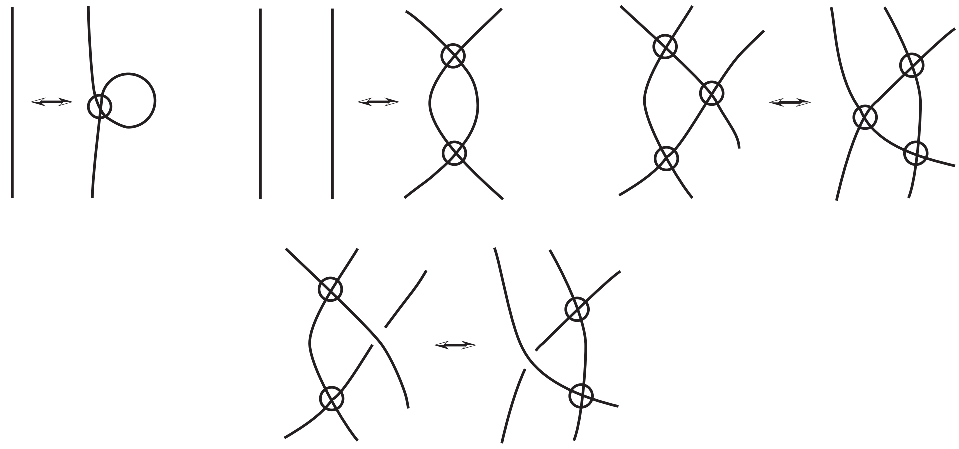

Figure 5: Virtual moves

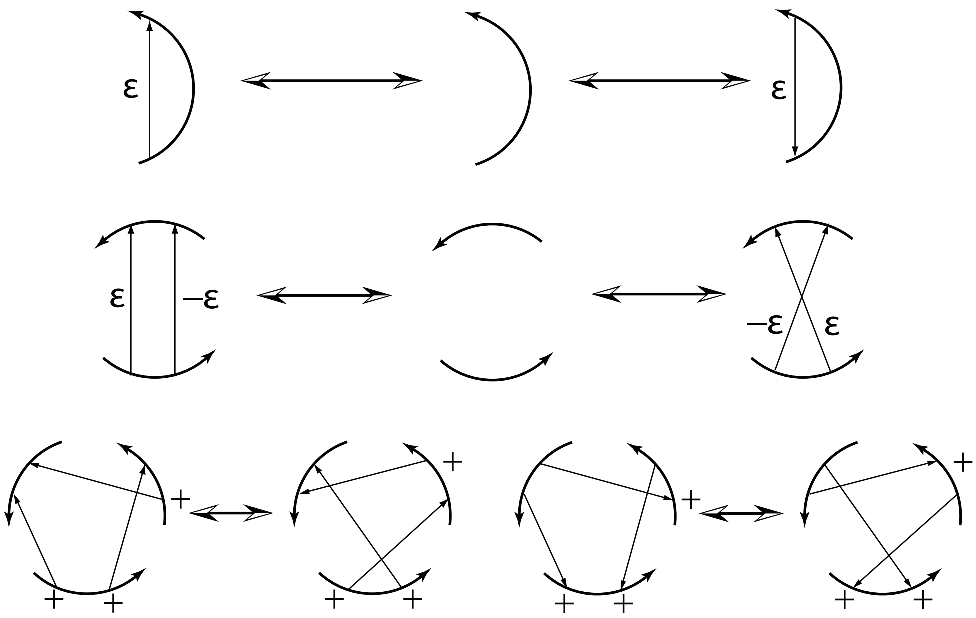

Figure 6: Moves in Gauss diagrams corresponding to Reidemeister moves.

In Section 2, we consider the degree of a crossing in a virtual knot K and grade virtual knots. For each natural number n, we get the virtual knot $V_n(K)$ obtained from K by virtualizing all crossings whose degrees cannot be divided by n. We show that $V_n(K_1)$ and $V_n(K_2)$ are equivalent if two virtual knots K_1 and K_2 are equivalent. We also show that for a virtual knot K and $n \in \mathbb{N}$, there exists a virtual knot K' such that K is equivalent to $V_n(K')$. By using connected sum of virtual knots, we show that for two given virtual knots K_1 and K_2, there is a virtual knot K' such that K_i is equivalent to $V_{n_i}(K')$ for $i = 1, 2$ if neither $n_1|n_2$ nor $n_2|n_1$. In Section 3, we give an example of using the grading to distinguish virtual knots and give a grade tree diagram of a virtual knot.

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II. GRADING VIRTUAL KNOTS

Z. Cheng introduced an odd writhe polynomial by using Gauss diagrams and weights $\mathrm{Ind}(c)$ of odd crossings c. We will slightly modify the weight $\mathrm{Ind}(c)$.

Let c be a crossing of a virtual knot diagram K. For simplicity sake we will denote the chord of the Gauss diagram $G(K)$ corresponding to the crossing c by c too. For a chord c of a Gauss diagram, let r_+ and r_- be the numbers of positive chords and negative chords traversing c from left to right respectively and let l_+ and l_- be the numbers of positive chords and negative chords traversing c from right to left respectively. We define the *degree* $d(c)$ of a crossing c by the equation

$$d(c) = r_+ - r_- - l_+ + l_-.$$

Actually $d(c) = -\mathrm{Ind}(c)$. For example, in Figure 7 we see that $d(c_1) = -3$ since $r_+ = 0$, $r_- = 2$, $l_+ = 1$ and $l_- = 0$. Similarly we can see that $d(c_2) = -2$, $d(c_3) = 0$ and $d(c_4) = -1$.

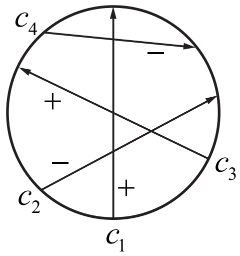

Figure 7

For a virtual knot diagram K, we denote the set of crossings of K by $C(K)$. For a natural number n, we define the set $C_n(K)$ of crossings of K as following

$$C_n(K) = \{c \in C(K) | d(c) \text{ is a multiple of } n\}.$$

By virtualizing all crossings of K whose degrees are not multiples of n, we get a virtual knot $V_n(K)$. For $n_1, \cdots, n_m \in \mathbb{N}$, $K(n_1, \cdots, n_m)$ is inductively defined as

$$\begin{cases} K(n_1) = V_{n_1}(K), \\ K(n_1, \cdots, n_m) = V_{n_m}(K(n_1, \cdots, n_{m-1})). \end{cases}$$

Let two virtual knot diagrams K_1 and K_2 be related by a Reidemeister move. Assume that d is a crossing of K_1 not involved with the Reidemeister move, then we denote the crossing of K_2 corresponding to d by d too. Similarly, in the Gauss diagrams $G(K_1)$ and $G(K_2)$, we use the same notation for the corresponding chords which are not involved with the Reidemeister move.

Now the polynomial $P_K(t)$ associated to K is defined by the equation

$$P_K(t) = \sum_{c \in C(K)} s(c)(t^{d(c)} - 1).$$

Theorem 2.1. [1, 9] *$P_K(t)$ is invariant under the Reidemeister moves.*

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We will show that $V_n(K_1)$ and $V_n(K_2)$ are equivalent if two virtual knot diagrams K_1 and K_2 are related by Reidemeister moves.

Lemma 2.2. *Let two virtual knot diagrams K_1 and K_2 be related by a first Reidemeister move. Then $V_n(K_1)$ and $V_n(K_2)$ are equivalent for all $n \in \mathbb{N}$.*

Proof. Suppose that K_1 and K_2 differ by a first Reidemeister move and their Gauss diagrams are related as shown in Figure 8 or Figure 9. If $d \in C(K_1) - \{c\}$ then the degree of d coincides with that of the corresponding crossings of K_2. Since the degree of the chord c in the Figures is zero, c belongs to $C_n(K_1)$. Then we see that $V_n(K_1)$ and $V_n(K_2)$ are also related by a first Reidemeister move.

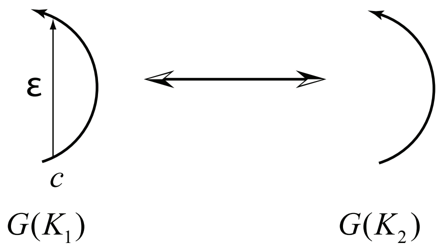

Figure 8

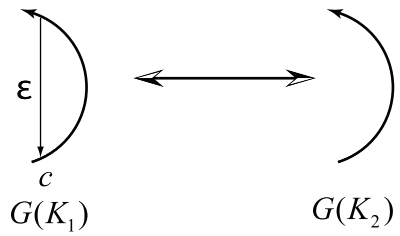

Figure 9

In [11], M. Polyak introduced four versions of the oriented second Reidemeister move as in Figure 10. The $\Omega 2a$ and $\Omega 2b$ moves can be represented as the move of Gauss

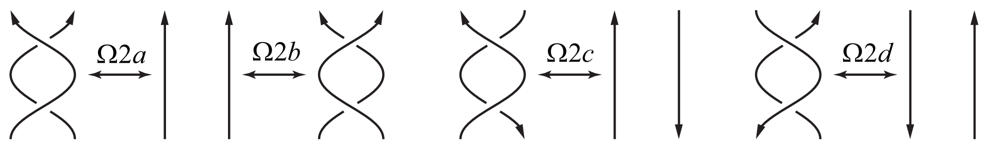

Figure 10

diagrams in Figure 11.

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Lemma 2.3. *Let K_1 and K_2 be related by a second Reidemeister move. Then $V_n(K_1)$ and $V_n(K_2)$ are equivalent for all $n \in \mathbb{N}$.*

Proof. Suppose that K_1 and K_2 are related by the second Reidemeister move and their Gauss diagrams are as shown in Figure 11. If $d \in C(K_1) - \{c_1, c_2\}$ then the degree of d coincides with that of the corresponding crossing of K_2. Since $d(c_1) = d(c_2)$, we see that $c_1 \in C_n(K_1)$ if and only if $c_2 \in C_n(K_1)$. Hence $V_n(K_1)$ and $V_n(K_2)$ are equivalent. Similarly, we can show that it holds for the other versions of the oriented second Reidemeister move.

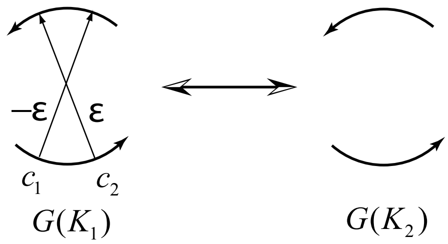

Figure 11

Lemma 2.4. *Let K_1 and K_2 be related by a third Reidemeister move. Then $V_n(K_1)$ and $V_n(K_2)$ are equivalent for all $n \in \mathbb{N}$.*

Proof. Suppose that K_1 and K_2 are related by the third Reidemeister move and their Gauss diagrams are related as shown in Figure 12. If

$$d \in C(K_1) - \{c_1, c_2, c_3\}$$

then the degree of d coincides with that of the corresponding crossings of K_2. So d belongs to $C_n(K_1)$ if and only if the corresponding crossing d' belongs to $C_n(K_2)$.
By using the mathematical induction for the number of crossings, we can see that

$$d(c_1) = d(c_2) + d(c_3)$$

and

$$d(c_i) = d(c_i')$$

for $i = 1, 2, 3$. So the number $|\{c_1, c_2, c_3\} \bigcap C_n(K_1)|$ of elements in the set $\{c_1, c_2, c_3\} \bigcap C_n(K_1)$ is either 1 or 3.
If

$$|\{c_1, c_2, c_3\} \bigcap C_n(K_1)| = 1$$

then

$$|\{c_1', c_2', c_3'\} \bigcap C_n(K_2)| = 1$$

and $V_n(K_1)$ and $V_n(K_2)$ are equivalent.
If

$$|\{c_1, c_2, c_3\} \bigcap C_n(K_1)| = 3$$

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then

$$|\{c_1', c_2', c_3'\} \bigcap C_n(K_2)| = 3$$

and $V_n(K_1)$ and $V_n(K_2)$ are related by the third Reidemeister move. So $V_n(K_1)$ and $V_n(K_2)$ are equivalent.

Similarly, we can show that it holds for the other versions of the oriented third Reidemeister move.

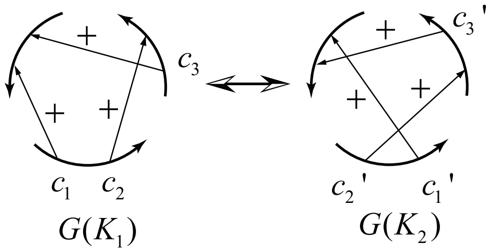

Figure 12

Combining Lemma 2.2, Lemma 2.3 and Lemma 2.4, we immediately get the following

Theorem 2.5. *If K_1 and K_2 are equivalent virtual knots then $V_n(K_1)$ and $V_n(K_2)$ are also equivalent for each $n \in \mathbb{N}$.*

By repeatedly applying Theorem 2.5 to virtual knots, we get the following

Corollary 2.6. *For natural numbers $n_1, \cdots, n_m$, if two virtual knots K_1 and K_2 are equivalent then $K_1(n_1, \cdots, n_m)$ and $K_2(n_1, \cdots, n_m)$ are also equivalent.*

Figure 13 illustrates that for a given virtual knot K how to construct a virtual knot K' such that K is equivalent to $V_3(K')$. We can add chords with degree ± 1 to change the degrees of the chords in K as desired. More generally we can get the following

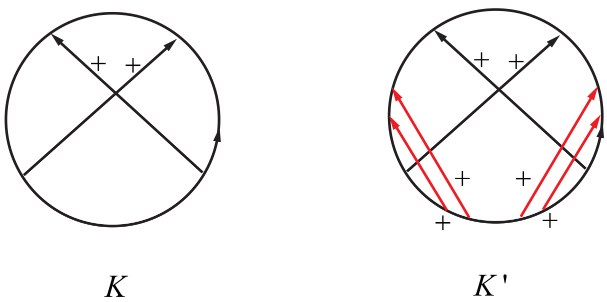

Figure 13.

Theorem 2.7. *For a virtual knot K and $n \in \mathbb{N}$, there exists a virtual knot K' such that K is equivalent to $V_n(K')$.*

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Table I

crossing	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
degree	-4	0	5	6	1	4	0	-1	-3

If we consider the connected sum of virtual knots we can also show the following

Theorem 2.8. *Assume that the greatest common divisor of n_1 and n_2 is neither n_1 nor n_2. For two given virtual knots K_1 and K_2, there is a virtual knot K' such that K_i is equivalent to $V_{n_i}(K')$ for $i = 1, 2$.*

III. EXAMPLES

Let L be the virtual knot as shown in Figure 14. Then it has the trivial affine index polynomial. But $V_2(L)$ is a non-trivial virtual knot with $P_{V_2(L)}(t) = (t-1)+(t^{-1}-1) \neq P_O(t)$, where O is the trivial knot. So we see that L is non-trivial.

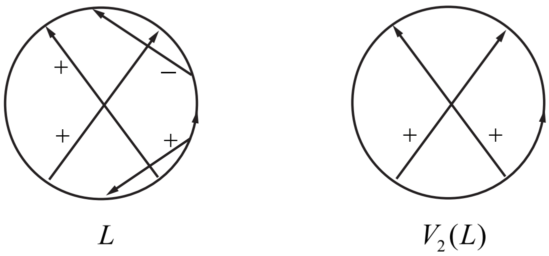

Figure 14

Let K be the virtual knot with nine crossings whose Gauss diagram is as shown in Figure 15.

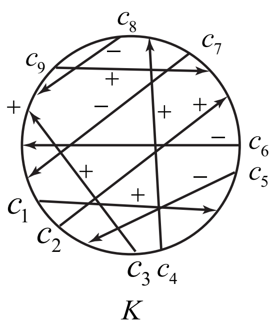

Figure 15

If we calculate the degree of crossings of K then we can get the Table 1.
Then we may get the grade tree diagram as shown in Figure 16.

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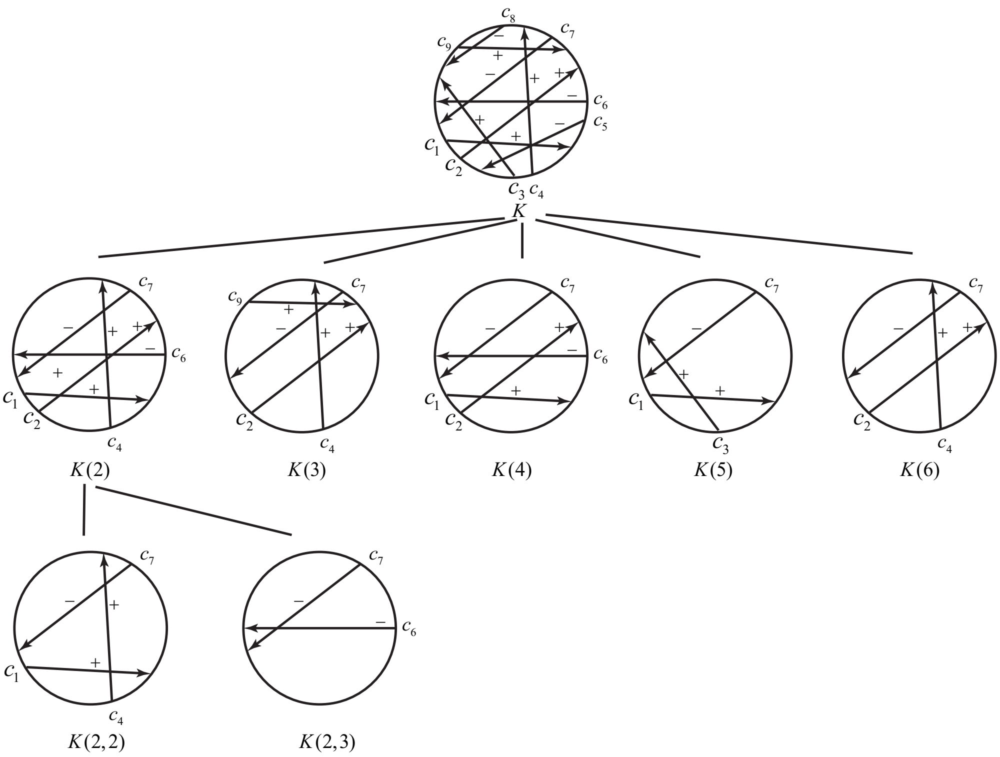

Figure 16

ACKNOWLEDGEMENT

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