A New Conformable Fractional Derivative in Generalized Functions

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ABSTRACT

In this article, we introduce an approach to fractional derivatives in the theory of generalized functions (Colombeau algebra G) using the new definition of the fractional derivative called "A New Conformable Fractional Derivative and Applications " introduced in [1].

$$(D^{\alpha} f)(t) = \lim_{h\to 0} \frac{f(t+he^{(\alpha-1)t}) - f(t)}{h},$$

for all t > 0, and a 2 (0;1).

We are going to show that if .f is an element of the Colombeau algebra then(Da f)is too, as well as the integral Ia f linked to this fractional derivation and we have introduced the important remark which supports and reinforces our new definition.

Keywords: conformable; fractional; derivative; generalized functions.

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In this article, we introduce an approach to fractional derivatives in the theory of generalized functions (Colombeau algebra G) using the new definition of the fractional derivative called "A New Conformable Fractional Derivative and Applications " introduced in [1].

$$(D^{\alpha} f)(t) = \lim_{h\to 0} \frac{f(t+he^{(\alpha-1)t}) - f(t)}{h},$$

for all t > 0, and a 2 (0;1).
We are going to show that if f is an element of the Colombeau algebra then (Da f)is too, as well as the integral Ia f linked to this fractional derivation and we have introduced the important remark which supports and reinforces our new definition.

Keywords: conformable; fractional; derivative; generalized functions.

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I. INTRODUCTION

In this work we give the definition of fractional conformal derivatives and their fractional integral associated with the sense [1] of order 0 < a < 1 in algebra of colombeau G [R+]. we proved this approach in the dove algebra in paper [2] using the derivative introduced by the author khalil in [5]. We notice that this conformal derivative has a defect concerning the fact that the derivatives of order k do not belong to G moreover if we reason by its regularized form: D then a difficulty will arise at the level of the composite of the fractional derivative and the associated integral and to remove this major difficulty, we will propose to study the Cauchy problem (3.1) with a conformal fractional derivative introduced by the authors ahmed chafiki and ahmed kajouni instead of fractional derivative compliant introduced in [5]. The object of fractional calculus is to generalize traditional derivatives to non-integer orders see [8, 6, 7, 3]. As is well known, many dynamical systems are best characterized by a dynamical fractional order model, usually based on the notion of noninteger order differentiation or integration.

The study of fractional order systems is more delicate than that of their integer order counterparts. Indeed, the fractional systems are, on the one hand, considered as memory systems, in particular to take account of the initial conditions and on the other hand they present a much more complex dynamics.

The theory of the fractional derivative is a very old theory, which goes back to a conversation of September 30, 1695 between Hˆopital and Leibniz concerning the definition of the operator. Dn dxn for n = 1 2 . Thus, over time certain approaches have been given in the literature such as the definition of Riemann-Liouville and that of Caputo.

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II. PRELIMINARIES AND NOTATIONS

2.1. Function Generalized

We use the following notations

$$\mathscr{A}_q = \{\varphi \in \mathscr{D}(\mathbb{R}^+) / \int_{\mathbb{R}^+} \varphi(x)dx = 1, \quad \int_{\mathbb{R}^+} x^\alpha \varphi(x)dx = 0$$

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$$\text{for} \quad 1 \leq |\alpha| \leq q\}$$

$q = 1, 2, \ldots$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon}) \quad \text{for} \quad \varphi \in \mathscr{D}(\mathbb{R}^+)$$

We denote by

$$\mathscr{E}(\mathbb{R}^+) = \{u : \mathscr{A}_1 \times \mathbb{R}^+ \to \mathbb{C} / \quad \text{with} \quad u(\varphi, x) \quad \text{is} \quad \mathscr{C}^\infty$$

$$\text{to the second variable} \quad x\}$$

$$u(x, \varphi_\varepsilon) = u_\varepsilon(x) \quad \forall \varphi \in \mathscr{A}_1$$

$$\begin{aligned} \mathscr{E}_M(\mathbb{R}^+) &= \{(u_\varepsilon)_{\varepsilon>0} \subset \mathscr{E}(\mathbb{R}^+) / \forall K \subset\subset \mathbb{R}^+, \forall m \in \mathbb{N}, \exists N \in \mathbb{N} \\ \text{such that} & \quad \sup_{x\in K} |\frac{d^m}{dx^m} u_\varepsilon(x)| \leq C\varepsilon^{-N} \quad \text{there exist constants} C > 0\} \end{aligned}$$

$$\begin{aligned} \mathscr{N}(\mathbb{R}^+) &= \{(u_\varepsilon)_{\varepsilon>0} \subset \mathscr{E}(\mathbb{R}^+) / \forall K \subset\subset \mathbb{R}^+, \forall m \in \mathbb{N}, \forall p \in \mathbb{N} \\ \text{such that} & \quad \sup_{x\in K} |\frac{d^m}{dx^m} u_\varepsilon(x)| \leq C\varepsilon^{p} \quad \text{there exist constants} C > 0\} \end{aligned}$$

The Colombeau algebra is defined as a factor set $\mathscr{G}(\mathbb{R}^+) = \mathscr{E}_M(\mathbb{R}^+)/\mathscr{N}(\mathbb{R}^+)$, where the elements of the set $\mathscr{E}_M(\mathbb{R}^+)$ are moderate while the elements of the set $\mathscr{N}(\mathbb{R}^+)$ are negligible.

2.2 Fractional Derivative

Definition 2.1.[1] Given a fonction $f : [0, \infty) \to \mathbb{R}$, then the conformable fractional derivative of f order α is defined by:

$$(D^\alpha f)(t) = \lim_{h\to 0} \frac{f(t + he^{(\alpha-1)t}) - f(t)}{h},$$

for all $t > 0$, and $\alpha \in (0, 1)$.
If f is α-differentiable in somme $(0, a), a > 0$ and $\lim_{t\to 0^+} (D^\alpha f)(t)$ exists, then define

$$(D^\alpha f)(0) = \lim_{t\to 0^+} (D^\alpha f)(t).$$

Theorem 2.2.[1] *Let $0 < \alpha \leq 1$ and f, g be α-differentiable at a point $t > 0$. Then*

(1) $D^\alpha(af + bg) = a(D^\alpha f) + b(D^\alpha g)$, *for all* $a,b \in \mathbb{R}$.
(2) $D^\alpha(t^p) = pe^{(\alpha-1)t}t^{p-1}$ *for all* $p \in \mathbb{R}$.
(3) $D^\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(4) $(D^\alpha fg) = f(D^\alpha g) + g(D^\alpha f)$.
(5) $(D^\alpha \frac{f}{g}) = \frac{f(D^\alpha g)+g(D^\alpha f)}{g^2}$.
(6) If in addition, f is differentiable, then $(D^\alpha f)(t) = e^{(\alpha-1)t} f'(t)$.

Theorem 2.3 [1] *Let $b,\ c,\ p \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then we have the following results*

(1) $D^{\alpha}(t^{p}) = pe^{(\alpha-1)t})t^{p-1}$ *for all* $p \in \mathbb{R}$.
(2) $D^{\alpha}(1) = 0$.
(3) $D^{\alpha}(e^{cx}) = ce^{(\alpha-1)x}e^{cx}$ *, for all* $c \in \mathbb{R}$.
(4) $D^{\alpha}(\sin(bx)) = be^{(\alpha-1)x}\cos(bx),\ b \in \mathbb{R}$.
(5) $D^{\alpha}(\cos(bx)) = -be^{(\alpha-1)x}\sin(bx),\ b \in \mathbb{R}$.

Theorem 2.4. [1] *However, it is worth noting the following conformable fractional derivatives of certain functions:*

(1) $D^{\alpha}(\sin(\frac{1}{1-\alpha}e^{(1-\alpha)t})) = \cos(\frac{1}{1-\alpha}e^{(1-\alpha)t})$.
(2) $D^{\alpha}(\cos(\frac{1}{1-\alpha}e^{(1-\alpha)t}))) = -\sin(\frac{1}{1-\alpha}e^{(1-\alpha)t}))$.
(3) $D^{\alpha}(e^{\frac{1}{1-\alpha}e^{(1-\alpha)t}}) = e^{\frac{1}{1-\alpha}e^{(1-\alpha)t}}$.
(4) $D^{\alpha}(\frac{1}{1-\alpha}e^{(1-\alpha)t}) = 1$.

2.3 Fractional Integral

As in the works of [5], it is interesting to note that, in spite of the variation of the definitions of the fractional derivatives, we can still adopt the same definition of the fractional integral here due to the fact that we obtained similar results in Theorem 2.5 as of the results (1)-(6) and (i)-(iii) in [5]. So, we have the following definition.

Definition 2.5.[1]

Let $\alpha \in (0,1)$ and $a \geq 0$, let f be a function defined on $(a,t]$, Then, the α -fractional integral of f is defined by,

$$I_{\alpha}^{a}(f)(t) = \int_{a}^{t} e^{(1-\alpha)s} f(s)ds.$$

Theorem 2.6. [1] *where $f : [a,\infty) \to \mathbb{R}$ is any continuous function in the domain of I_{α} and $0 < \alpha \leq 1$. Then, for $t > a$ we have*

$$D_{a}^{\alpha} I_{\alpha}^{a} f(t) = f(t).$$

Theorem 2.7. *Let $f : (a,b) \to \mathbb{R}$ is function differentiable and $0 < \alpha < 1$. Then, for $t > a$, we have*

$$I_{\alpha}^{a} D_{a}^{\alpha} f(t) = f(t) - f(a).$$

III. FRACTIONAL DERIVATIVES OF COLOMBEAU ALGEBRA

Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of a Colombeau generalized $u \in \mathscr{G}([0,\infty))$. The fractional derivative of $(u_{\varepsilon})_{\varepsilon>0}$, is defined by

$$D^{\alpha}u_{\varepsilon}(x) = e^{(\alpha-1)x}\frac{d}{dx}u_{\varepsilon}(x)$$

Lemma 3.1. *Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of $u \in \mathscr{G}([0,\infty))$. Then, for every $0 < \alpha < 1$, $\sup_{x\in[0,T]}\left|D^{\alpha}u_{\varepsilon}(x)\right|$ has a moderate bound.*

Proof.

$$\begin{aligned}\sup_{x\in[0,T]}\left|D^{\alpha}u_{\varepsilon}(x)\right| = \sup_{x\in[0,T]}\left|e^{(\alpha-1)x}\frac{d}{dx}u_{\varepsilon}(x)\right| &\leq C\sup_{x\in[0,T]}\left|\frac{d}{dx}u_{\varepsilon}(x)\right| \\ &\leq \tilde{C}\varepsilon^{-N}\end{aligned}$$

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Lemma 3.2. *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u \in \mathscr{G}([0,\infty))$. Then, for every $0<\alpha<1$,$\sup_{x\in[0,T]}\left|D^{\alpha}u_{1,\varepsilon}(x)-D^{\alpha}u_{2,\varepsilon}(x)\right|$ is negligible.*

Proof.

$$\begin{aligned}
\sup_{x\in[0,T]}\left|D^{\alpha}u_{1,\varepsilon}(x)-D^{\alpha}u_{2,\varepsilon}(x)\right| &= \sup_{x\in[0,T]}\left|e^{(\alpha-1)x}\frac{d}{dx}u_{1,\varepsilon}(x)-e^{(\alpha-1)x}\frac{d}{dx}u_{2,\varepsilon}(x)\right| \\
&= \sup_{x\in[0,T]}\left|e^{(\alpha-1)x}(\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x))\right| \\
&\leq C\sup_{x\in[0,T]}\left|\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x)\right|
\end{aligned}$$

Since $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$represent the same Colombeau generalized u we have that $\sup_{x\in[0,T]}\left|\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x)\right|$ is negligible.Therefore,$\sup_{x\in[0,T]}\left|D^{\alpha}u_{1,\varepsilon}(x)-D^{\alpha}u_{2,\varepsilon}(x)\right|$ is negligible, too. □

Lemma 3.3. *Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of $u \in \mathscr{G}([0,\infty))$. Then, for every $0<\alpha<1$ and every $k\in\{0,1,2,\ldots\}$,$\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}D^{\alpha}u_{\varepsilon}(x)\right|$ has a moderate bound.*

Proof. For arbitrary order derivative, we have

$$\begin{aligned}
\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}D^{\alpha}u_{\varepsilon}(x)\right| &= \sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}e^{(\alpha-1)x}\frac{d}{dx}u_{\varepsilon}(x)\right| \\
&\leq C_1\sup_{s\in K}\left|\frac{d}{dx}u_{\varepsilon}(x)\right|+C_2\sup_{s\in K}\left|\frac{d^2}{dx^2}u_{\varepsilon}(x)\right|+\ldots+C_{k+1}\sup_{s\in K}\left|\frac{d^{k+1}}{dx^{k+1}}u_{\varepsilon}(x)\right|
\end{aligned}$$

$k\in\mathbb{N}$ for some constant $C>0$

□

according to $\sup_{s\in K}\left|\frac{d^k}{dx^k}u_{\varepsilon}(x)\right|$ has a moderate bound,for every $k\in\{0,1,2,\ldots\}$.
Therefore $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}D^{\alpha}u_{\varepsilon}(x)\right|$ has a moderate bound,too.

Lemma 3.4. *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u \in \mathscr{G}([0,\infty))$. Then, for every $0<\alpha<1$,and every$k\in\{0,1,2,\ldots\}$, $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(D^{\alpha}u_{1,\varepsilon}(x)-D^{\alpha}u_{2,\varepsilon}(x))\right|$ is negligible.*
Proof.

$$\begin{aligned}
&\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(D^{\alpha}u_{1,\varepsilon}(x)-D^{\alpha}u_{2,\varepsilon}(x))\right| \\
&= \sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(e^{(\alpha-1)x}\frac{d}{dx}u_{1,\varepsilon}(x)-e^{(\alpha-1)x}\frac{d}{dx}u_{2,\varepsilon}(x))\right| \\
&= \sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}e^{(\alpha-1)x}(\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x))\right| \\
&\leq C\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}e^{(\alpha-1)x}(\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x))\right| \\
&\leq C_1\sup_{x\in[0,T]}\left|(\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x))\right|+C_2\sup_{x\in[0,T]}\left|\frac{d^2}{dx^2}(u_{1,\varepsilon}(x)-\frac{d^2}{dx^2}u_{2,\varepsilon}(x))\right|+\ldots+C_{k+1}\sup_{x\in[0,T]}\left|\frac{d^{k+1}}{dx^{k+1}}(u_{1,\varepsilon}(x)-\frac{d^{k+1}}{dx^{k+1}}u_{2,\varepsilon}(x))\right| \\
&\leq C_1\sup_{x\in[0,T]}\left|(\frac{d}{dx}(u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x)))\right|+C_2\sup_{x\in[0,T]}\left|\frac{d^2}{dx^2}(u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x))\right|+\ldots+C_{k+1}\sup_{x\in[0,T]}\left|\frac{d^{k+1}}{dx^{k+1}}(u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x))\right|
\end{aligned}$$

as $\sup_{s\in K}\left|\left(\frac{d^k}{dx^k}u_{1,\varepsilon}-\frac{d^k}{dx^k}u_{2,\varepsilon}\right)(s)\right|$ is negligible then $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(D^{\alpha}u_{1,\varepsilon}(x)-D^{\alpha}u_{2,\varepsilon}(x))\right|$ is negligible. Now we introduce the fractional derivative of a Colombeau generalized on $[0,\infty)$ in the following way.

Definition 3.5. Let $u\in\mathscr{G}([0,\infty))$ be a Colombeau generalized on $[0,\infty)$ The α th fractional derivative of u, in notation $D^{\alpha}u(x)=[D^{\alpha}u_{\varepsilon}(x)]$,is the element of $\mathscr{G}([0,\infty))$ satisfying (3).

IV. FRACTIONAL INTEGRAL OF COLOMBEAU GENERALIZED

Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of a Colombeau generalized $u\in\mathscr{G}([0,\infty))$. The fractional integral of $(u_{\varepsilon})_{\varepsilon>0}$, is defined by

$$I^{\alpha}u_{\varepsilon}(x)=\int_0^x(e^{(1-\alpha)s}u_{\varepsilon}(s))ds \tag{4.1}$$

Lemma 4.1. *Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of $u\in\mathscr{G}([0,\infty))$. Then, for every $0<\alpha<1$,$\sup_{x\in[0,T]}\left|I^{\alpha}u_{\varepsilon}(x)\right|$ has a moderate bound.*

Proof.

$$\begin{aligned}
\sup_{x\in[0,T]}\left|I^{\alpha}u_{\varepsilon}(x)\right| &= \sup_{x\in[0,T]}\left|\int_0^x(e^{(1-\alpha)s}u_{\varepsilon}(s))ds\right|\\
&\leq \sup_{s\in[0,T]}\left|u_{\varepsilon}(s)\right|\sup_{x\in[0,T]}\left|\int_0^x(e^{(1-\alpha)s})ds\right|\\
&\leq \frac{e^{-T}-1}{1-\alpha}\sup_{s\in[0,T]}\left|u_{\varepsilon}(s)\right|\\
&\leq C_{\alpha,T}\varepsilon^{-N}
\end{aligned}$$

Lemma 4.4. *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u\in\mathscr{G}([0,\infty))$. Then, for every $0<\alpha<1$,$\sup_{x\in[0,T]}\left|I^{\alpha}u_{1,\varepsilon}(x)-I^{\alpha}u_{2,\varepsilon}(x)\right|$ is negligible.*

Proof.

$$\begin{aligned}
\sup_{x\in[0,T]}\left|I^{\alpha}u_{1,\varepsilon}(x)-I^{\alpha}u_{2,\varepsilon}(x)\right| &= \sup_{x\in[0,T]}\left|\int_0^x(e^{(1-\alpha)s}u_{1,\varepsilon}(s))ds-\int_0^x(s^{\alpha-1}u_{2,\varepsilon}(s))ds\right|\\
&= \sup_{x\in[0,T]}\left|\int_0^x(e^{(1-\alpha)s}(u_{1,\varepsilon}(s)-u_{2,\varepsilon}(s))ds\right|\\
&\leq \sup_{s\in[0,T]}\left|u_{1,\varepsilon}(s)-u_{2,\varepsilon}(s)\right|\left|\int_0^x(e^{(1-\alpha)s})ds\right|\\
&\leq \frac{e^{-T}-1}{1-\alpha}\sup_{s\in[0,T]}\left|u_{1,\varepsilon}(s)-u_{2,\varepsilon}(s)\right|\bigg|
\end{aligned}$$

Since $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$represent the same Colombeau generalized u we have that

$$\sup_{x\in[0,T]}\left|u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x)\right|$$

is negligible.Therefore,

$$\sup_{x\in[0,T]}\left|I^{\alpha}u_{1,\varepsilon}(x)-I^{\alpha}u_{2,\varepsilon}(x)\right|$$

is negligible, too.

Lemma 4.3. *Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of $u\in\mathscr{G}([0,\infty))$.*
Then, for every $0<\alpha<1$ and every $k\in\{0,1,2,...\}$,

$$\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}I^{\alpha}u_{\varepsilon}(x)\right|$$

has a moderate bound.

Proof. Let $\varepsilon\in(0,1)$ $0<\alpha<1$, we have

$$\begin{aligned}\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}I^{\alpha}u_{\varepsilon}(x)\right| &= \sup_{x\in[0,T]}\left|\frac{d^{k-1}}{dx^{k-1}}e^{(\alpha-1)s}u_{\varepsilon}(x)\right| \\ &\leq C_0\sup_{s\in K}\left|u_{\varepsilon}(x)\right|+C_1\sup_{s\in K}\left|\frac{d}{dx}u_{\varepsilon}(x)\right|+... \\ &+ C_{k-1}\sup_{s\in K}\left|\frac{d^{k-1}}{dx^{k-1}}u_{\varepsilon}(x)\right|\end{aligned}$$

$k\in\mathbb{N}$ for some constant $C_i>0$,$i\in\{0,1,2,...,k-1\}$

Since, $\sup_{s\in K}\left|\frac{d^k}{dx^k}u_{\varepsilon}(x)\right|$ has a moderate bound, for every $k\in\{0,1,2,...\}$, it follows that, $\sup_{x\in[0,T]}\left|I^{\alpha}u_{\varepsilon}(x)\right|$ has a moderate bound, too.
For arbitrary order derivative.

Lemma 4.4. *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u\in\mathscr{G}([0,\infty))$.*
Then, for every $0<\alpha<1$,and every $k\in\{0,1,2,...\}$,

$$\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(I^{\alpha}u_{1,\varepsilon}(x)-I^{\alpha}u_{2,\varepsilon}(x))\right|$$

is negligible.

Proof.

$$\begin{aligned}\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(I^{\alpha}u_{1,\varepsilon}(x)-I^{\alpha}u_{2,\varepsilon}(x))\right| &= \sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}((I^{\alpha}u_{1,\varepsilon})(x)-(I^{\alpha}u_{2,\varepsilon})(x))\right| \\ &= \sup_{x\in[0,T]}\left|\frac{d^{k-1}}{dx^{k-1}}(e^{(\alpha-1)x}u_{1,\varepsilon}-e^{(\alpha-1)x}u_{2,\varepsilon})(x)\right| \\ &= \sup_{x\in[0,T]}\left|\frac{d^{k-1}}{dx^{k-1}}e^{(\alpha-1)x}(u_{1,\varepsilon}-u_{2,\varepsilon})(x)\right| \\ &\leq C_0\sup_{x\in[0,T]}\left|u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x)\right|+C_1\sup_{x\in[0,T]}\left|\frac{d}{dx}(u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x))\right|+ \\ &...+ C_{k-1}\sup_{x\in[0,T]}\left|\frac{d^{k-1}}{dx^{k-1}}(u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x))\right|\end{aligned}$$

since, $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x))\right|$ is negligible then $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(I^\alpha u_{1,\varepsilon}(x)-I^\alpha u_{2,\varepsilon}(x))\right|$ is negligible.
Now we introduce the fractional integral of a Colombeau generalized on $[0,\infty)$ in the following way.

Definition 4.5. Let $u\in\mathscr{G}([0,\infty))$ be a Colombeau generalized on $[0,\infty)$ The αth fractional integral of u, in notation

$$I^\alpha u(x) = [I^\alpha u_\varepsilon(x)]$$

is the element of $\mathscr{G}([0,\infty))$ satisfying (4.1).

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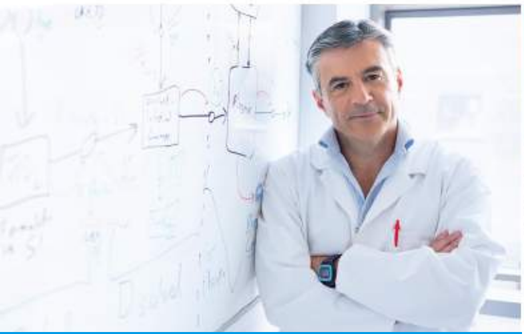

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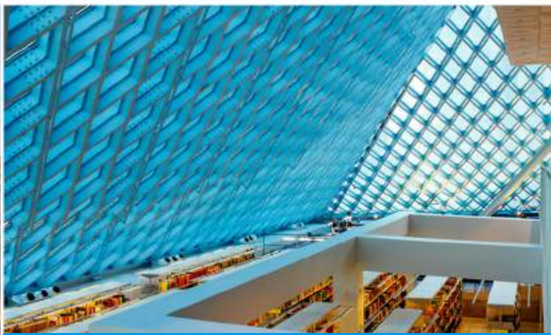

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