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Role of Time Delay on Sensitivity and Hopf-Bifurcation between two Competing Plant Populations under Allelopathy

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The study of dynamical interactions between two competing plant populations under the influence of allelochemicals is a current area of active research. In this paper, we have studied the effect of allelochemicals on the plant populations of two competing species model with particular emphasis on time reliant variations in their densities. Positivity, Boundedness, Equilibrium point is calculated. We have also studied the stability of the dynamics system about non-zero equilibrium points with the help of the Routh-Hurwitz theorem. The system shows asymptotic stability when the value of delay parameter is less than the critical value. Hopf-bifurcation occurs when the value of the delay parameter is greater than the critical value. In addition, we calculate the sensitivity analysis of the dynamics system to understand how the various sources of uncertainty in a model contributes to the overall uncertainty.

Keywords: allelopathy, sensitivity analysis, delay, hopf-bifurcation.

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I. INTRODUCTION

Allelopathic interference mechanisms are extremely difficult, if not impossible, to distinguish from interference due to competing in economic and environmental conditions. Plant interference is defined as any physical or chemical process that causes a plant's development to be radiated over time as a result of the availability of another plant. Riha and Bocharov investigate the descriptive and analytical relevance of delay in plant populations, metabolism, and other areas, as well as the sensitivity analysis in numerical modeling utilizing the delay differential equation, applying a direct method [1]. Riha also examined the sensitivity analysis for dynamic systems with delays utilizing the direct technique when the model's parameters are not constant but change over time [2]. Kalra and Kumar studies the role of delay in plant development under the effect of toxic metal [3]. Dipesh and Kumar worked on stimulatory and inhibitory allelopathy effects on two competing plant populations [4]. Gupta et al. studied the effect of allelopathic on competing harvesting species and observed that realizes allelochemicals are inhibitory for each other [5]. In a predator-prey model with discrete delays, a systematic analysis was presented for several types of stabilities such as absolute stability, conditionally stability, and bifurcation [6]. The nature of exponential characteristic equation zeros studied in depth was proposed by Wei and Ruan [7]. A Lotka-Volterra competitive system for two planktonic ecosystems in the presence of toxicity proposed by Chattopadhyay [8]. Monica et al. observed that allelopathy may also be considered a significant impact in nature crops when coping with cooperative or sequential development of a variety of plants [9]. The response of aquatic competitive models in the handling of toxic substances is a hot topic right now. The impact of toxic chemicals on the formation of algal communities is explored utilizing a variety of dynamics factors, including stability, elimination of both species, and delay differential equation systems. [10-14]. The impact of toxic effects on phytoplankton populations is being studied by aquatic ecological specialists [15-16]. Sun et al. suggested a response–

diffusion equation to show the interaction between plant transcriptional protections and herbivore transcriptional protections. They assumed that the plant harmed by the herbivore using transcriptional protections requires some response time, which may be expressed by time delay [17]. Kumar and Dipesh studied the effect of allelochemicals on competing plant populations using time lag [18].

In light of the above discussion, a mathematical method is developed for analyzing the role of delay in the allelopathic effect competition of a community of plants when one population releases allelochemicals and affects the other, while the other organism does not develop any allelochemicals and discovers that the affecting population may be pushed to death.

II. MATHEMATICAL MODEL

A set of nonlinear ordinary differential equations lead the classic two-species Lotka-Volterra type competition model.

$$\frac{dP_1}{dt} = P_1(c_1 - \alpha_1 P_1 - \beta_{12} P_2) \quad (1)$$

$$\frac{dP_2}{dt} = P_2(c_2 - \alpha_2 P_2 - \beta_{21} P_1) \quad (2)$$

Where P_1 and P_2 are the two competing plant populations species respectively, c_1 and c_2 are the cell proliferation rate of 1st and 2nd plant populations respectively, α_1 and α_2 are the intraspecific competition of 1st and 2nd plant population species. β_{12} and β_{21} are the interspecific competition of 1st and 2nd plant population species. The units of c_1, c_2, α_1 and α_2 is the number of days per cell and days is the unit of time.

We take an additional term into the Lotka-Volterra equation of second plant population species to the model for the allelochemicals interaction.

$$\frac{dP_1}{dt} = P_1(c_1 - \alpha_1 P_1 - \beta_{12} P_2) \quad (3)$$

$$\frac{dP_2}{dt} = P_2(c_2 - \alpha_2 P_2 - \beta_{21} P_1 + \gamma P_1 P_2) \quad (4)$$

γ denotes the role of allelochemicals substance realized by second plant population species and affects the first plant population with having all parameters have same biotic explanation above. Release of allelochemicals is not suddenly and but some discrete time delay is required for plant population to mature. This time delay is introduced in the release of allelochemicals by 1st plant population and introduces discrete delay into the intraspecific competition term of both plant populations. The competition model is defined for the system as:

$$\frac{dP_1}{dt} = P_1(c_1 - \alpha_1 P_1(t - \tau) - \beta_{12} P_2) \quad (5)$$

$$\frac{dP_2}{dt} = P_2(c_2 - \alpha_2 P_2(t - \tau) - \beta_{21} P_1 + \gamma P_1(t - \tau) P_2) \quad (6)$$

Where $P_1 \geq 0, P_2 \geq 0$ for all t & $P_1(t - \tau) \& P_2(t - \tau) = \text{Constant}$ for $t \in [0, \tau]$.

For understanding a biological phenomena we check the basis properties like positivity, boundedness, equilibrium and stability. These properties would make it easier to calculate the validity of the biological phenomena. Whereas delay occurs in nature and takes part in the system to understand the dynamics of biological phenomena.

2.1 Boundedness of Solution

Let $W = P_1 + P_2$

$$\frac{dW}{dt} = \frac{dP_1}{dt} + \frac{dP_2}{dt}$$

Let $\chi = (\alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \gamma)$ & $P_1(t - \tau) \cong P_1, P_2(t - \tau) \cong P_2$

$$\frac{dW(t)}{dt} \leq \chi(P_1 + P_2) - (c_1 + c_2)$$

$$\Rightarrow 0 \leq \chi W - (c_1 + c_2)$$

$$\Rightarrow 0 \leq W \leq \frac{(c_1 + c_2)}{\chi}$$

All solution of (3) – (4) lying in the region $S = \left[(P_1, P_2) \in R_+^2 : 0 \leq W \leq \frac{(c_1 + c_2)}{\chi} \right]$ as $t \rightarrow \infty \forall$ positively initial value $P_1 \geq 0, P_2 \geq 0 \forall t$ & $P_1(t - \tau), P_2(t - \tau) = \text{constant for } t \in [0, \tau] \in S \subset R_+^2$.

2.2 Positivity of Solution

All the given solution of (5) – (6) are positive with initial condition $P_1 \geq 0, P_2 \geq 0 \forall t$ & $P_1(t - \tau), P_2(t - \tau) = \text{constant for } t \in [0, \tau]$. The significance of (P_1, P_2) remain positive for all $t > 0$.

From Equ. (5)

$$\begin{aligned} \frac{dP_1}{dt} &\geq (c_1 - \alpha_1 P_1 - \beta_{12} P_2) P_1 \\ \Rightarrow \frac{dP_1}{P_1} &\geq \left(c_1 - \frac{(c_1 + c_2)}{\chi} (\alpha_1 - \beta_{12}) \right) dt \\ \Rightarrow P_1 &\geq e^{-\left(c_1 - \frac{(c_1 + c_2)}{\chi} (\alpha_1 - \beta_{12}) \right) t} \end{aligned}$$

Similarly we calculate $P_2, \forall P_1 \geq 0, P_2 \geq 0$.

2.3 Equilibrium Points for the Solution

Several equilibrium point for the dynamical system (5)-(6) are $E_{00}, E_{10}, E_{01}, E^*$ exist with no limitation on the variable of the dynamical system

$$E_{00} : (0, 0) \text{ (zero equilibrium point)}$$

$$E_{10} : \left(\frac{c_1}{\alpha_1}, 0 \right) \quad (\text{axial equilibrium point, unstable})$$

$$E_{01} : \left(0, \frac{c_2}{\alpha_2} \right) \quad (\text{axial equilibrium point, unstable})$$

$$E^* (P_1^*, P_2^*) \quad (\text{Non - zero equilibrium point, stable})$$

We study only non-zero equilibrium point $E^*(P_1^*, P_2^*)$

From Equ. (5)

$$\frac{dP_1^*}{dt} = P_1^* (c_1 - \alpha_1 P_1^* - \beta_{12} P_2^*)$$

$$P_1^* \neq 0, \text{ so } c_1 - \alpha_1 P_1^* - \beta_{12} P_2^* = 0$$

$$\text{We get } P_1^* = \frac{c_1 - \beta_{12} P_2^*}{\alpha_1} \quad (7)$$

and similarly we calculate $P_2^* = \frac{c_2}{\alpha_2 - \beta_{21} - \gamma P_1^*}$

Put the value of P_2^* in (7) we get a quadratic equation in P_1^*

$$\alpha_1 \gamma P_1^{*2} - (\gamma c_1 + \alpha_1 \alpha_2 + \alpha_1 \beta_{21}) P_1^* + c_1 \alpha_2 + c_1 \beta_{21} - \beta_{12} c_1 = 0$$

Which have two roots

$$P_{1,2}^* = \frac{(\gamma c_1 + \alpha_1 \alpha_2 + \alpha_1 \beta_{21}) \pm \sqrt{(\gamma c_1 + \alpha_1 \alpha_2 + \alpha_1 \beta_{21})^2 - 4 \alpha_1 \gamma (c_1 \alpha_2 + c_1 \beta_{21} - \beta_{12} c_1)}}{2 \alpha_1 \gamma} \quad (8)$$

Local Stability of Equilibrium Point (P_1^*, P_2^*) and Hopf-Bifurcation

$$\frac{dP_1^*}{dt} = P_1^* (c_1 - \alpha_1 P_1^*(t - \tau) - \beta_{12} P_2^*)$$

$$\frac{dP_2^*}{dt} = P_2^* (c_2 - \alpha_2 P_2^*(t - \tau) - \beta_{12} P_2^* + \gamma P_1^*(t - \tau) P_2^*)$$

The characteristic equation of the system is

$$A(\lambda, \tau) = \lambda^2 + (a_1 e^{-\lambda \tau} + a_2) \lambda + b_1 + e^{-\lambda \tau} d = 0 \quad (9)$$

Where $a_1 = 2\gamma P_2^* - 2\alpha_2 P_2^* - 2\alpha_1 P_1^*$

$$a_2 = 3\beta_{12} P_2^* - c_1 - c_2$$

$$b_1 = c_1 c_2 - 2\beta_{12} P_2^* c_1$$

$$d = c_1 + c_2 + c_3 + c_4$$

Where $c_1 = 2P_2^*(c_1\alpha_2 - \beta_{12}\alpha_2 P_2^*)$, $c_2 = 2\alpha_1 P_1^*(c_2 - 2\beta_{12} P_2^* - 2)$, $c_3 = -2\alpha_2 P_2^*$

$$c_4 = \gamma P_2^*(2c_1 + 2\beta_{12} P_2^* - 2 + \beta_{12} P_1^*)$$

Clearly a_1, a_2, b_1 , and d all are positive.

If we put $\tau = 0$ in Equ. (9), we get

$$\lambda^2 + (a_1 + a_2)\lambda + b_1 + d = 0 \quad (10)$$

With the help of Routh-Hurwitz criteria, root of Equ. (10) will be negative real part, i.e., the dynamical system is stable iff:

$$(g_1): (a_1 + a_2) > 0, (g_2): (b_1 + d) > 0 \text{ Which is true.}$$

At $\tau = 0$, E^* is locally asymptotically stable when $\frac{\alpha_1}{\beta_{12}} > \max\left[\frac{\beta_{21}}{\alpha_2}, \frac{P_1^*}{P_2^*}\right]$

E^* will be locally asymptotically stable for $\tau \geq 0$ when the real part of $A(\lambda, \tau)$ is -ve and $A(i\omega, \tau) \neq 0$ for every real ω and $\tau \geq 0$. Now we calculate the negative real component of the root moves for the positive real component root when the value of τ is different.

Assume $\lambda = i\phi$, $\phi > 0$ be the root of Equ. (9), then (9) became

$$(i\phi)^2 + (a_1 e^{-i\phi\tau} + a_2)(i\phi) + b_1 + e^{-i\phi\tau} d = 0$$

$$\Rightarrow -\phi^2 + (a_1(\cos\phi\tau - i\sin\phi\tau) + a_2)(i\phi) + b_1 + d(\cos\phi\tau - i\sin\phi\tau) = 0$$

Separate real and imaginary part, we get

$$\phi^2 - b_1 = a_1\phi\sin\phi\tau + d\cos\phi\tau \quad (11)$$

$$a_2\phi = d\sin\phi\tau - a_1\phi\cos\phi\tau \quad (12)$$

Squaring and adding (11) and (12), we get

$$\phi^4 + (a_2^2 - 2b_1 - a_1^2)\phi^2 + b_1^2 - d^2 = 0 \quad (13)$$

Equ. (13) have two roots

$$\phi_{1,2}^2 = \frac{(a_2^2 + 2b_1 - a_1^2) \pm \sqrt{(a_2^2 - 2b_1 - a_1^2)^2 - 4(b_1^2 - d^2)}}{2} \quad (14)$$

No roots of $\phi_{1,2}^2$ is +ve if:

$$(g_3): (a_1^2 + 2b_1 - a_2^2) < 0 \text{ and } (b_1^2 - d^2) > 0 \text{ or } (a_2^2 - 2b_1 - a_1^2)^2 < 4(b_1^2 - d^2)$$

So, Equ. (14) doesn't have positive root if condition (g_3) hold.

Lemma 1. Root of Equ. (9) have negative real part for all $\tau \geq 0$, if $(g_1) - (g_2)$ hold.

Other side if

$$(g_4): (b_1^2 - d^2) < 0 \text{ or } (a_1^2 + 2b_1 - a_2^2) > 0 \text{ and } (a_2^2 - 2b_1 - a_1^2)^2 = 4(b_1^2 - d^2)$$

It shows that Equ. (11) have one positive root, which is ϕ_1^2 . If

$(g_5): (b_1^2 - d^2) > 0 \text{ or } (a_1^2 + 2b_1 - a_2^2) > 0 \text{ and } (a_2^2 - 2b_1 - a_1^2)^2 > 4(b_1^2 - d^2)$ Which shows that Equ. (11) have two positive roots, which is $\phi_{1,2}^2$.

In (g_4) and (g_5) , when we change the value of τ , root of Equ. (9) are purely imaginary. With the help of Equ. (11)-(12) critical value of τ_i^\pm of τ are calculated :

$$\tau_i^\pm = \frac{1}{\phi_{1,2}} \cos^{-1} \frac{(\phi_{1,2}^2 - b_1)}{d} + \frac{2i\pi}{\phi_{1,2}}, i = 0, 1, 2, \dots \tag{15}$$

The above discussion can be condensed in succeeding by S. Raun [5].

Lemma 2. (i) Equ. (9) has a pair of imaginary roots $\pm i\phi_1$, if $(g_1) - (g_2)$ and (g_4) hold and $\tau = \tau_i^+$.

(ii) Equ. (9) has a pair of imaginary roots $\pm i\phi_2 (\pm i\phi_1)$ respectively, if $(g_1) - (g_2)$ and (g_5) hold and $\tau = \tau_i^- (\tau = \tau_i^+ \text{ resp.})$.

Our exception is the negative real part shifted toward positive real part when $\tau > \tau_i^+$ & $\tau < \tau_i^+$. We check the possibilities

$$\tau_i^\pm = \mu_i^\pm(\tau) + i\phi_i^\pm(\tau); i = 0, 1, 2, 3, \dots$$

The roots of Equ. (9) satisfied. $\mu_i^\pm(\tau_i^\pm) = 0, \phi_i^\pm(\tau_i^\pm) = \phi_{1,2}$

We can verify that the following transversely conditions

$$\frac{d}{d\tau} (Re\lambda_i^+(\tau_i^+)) > 0 \text{ and } \frac{d}{d\tau} (Re\lambda_i^-(\tau_i^-)) < 0$$

It shows that τ_i^\pm varies with different values. The distribution of the Equ. (9) zeros are determined by the S. Raun,

Theorem 1. Let τ_l^+ ($l = 0, 1, 2, 3\dots$) be defined by Equ. (15).

(1) If $(g_1), (g_2)$ hold, then all of root (9) have -ve real part $\forall \tau \geq 0$.

(2) If $(g_1), (g_2)$ and (g_4) hold and when $\tau \in [0, \tau_0^+)$, then all of root (9) have negative real parts. When $\tau = \tau_0^+$, then (9) has a pair of purely imaginary roots $\pm i\phi_1$. When $\tau > \tau_0^+$, (9) has at least one +ve real part root.

(3) If $(g_1), (g_2)$ and (g_5) carry & +ve integer n s.t. $0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- \dots \dots < \tau_{n-1}^- < \tau_n^+$ and there is n oscillation from stable to unstable. Which shows $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \dots \dots (\tau_{n-1}^+, \tau_n^+)$ all the roots of Equ. (9) have -ve real parts. When $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \dots \dots (\tau_{n-1}^+, \tau_n^+)$ & $\tau > \tau_n^+$, Equ. (9) has at least one root with real parts.

Sensitivity Analysis of State variables w.r.t. model parameter

The model contains constant parameters in this article. The 'Direct Method' is used for estimating the coefficients of generalized sensitivity. The direct approach is based on assuming all parameters as fixed then somehow, by utilizing the sensitivity equations with initial solution of equations, the sensitivity coefficients are calculated. When all the parameters $\alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \gamma$ in the proposed system (5) - (6) are assumed to be constant, then sensitivity analysis may simply involve finding the solution's partial derivatives with regard to each parameter in this case. As an example, if parameter c_1 considered then partial derivatives of the solution (P_1, P_2) w.r.t. c_1 gives rise to following set of sensitivity equation:

$$\frac{dS_1}{dt} = S_1 P_1 - \beta_{12}(S_2 P_1 - S_1 P_2) - 2\alpha_1 P_1(t - \tau) S_1(t - \tau) \tag{16}$$

$$\frac{dS_2}{dt} = S_2 P_2 - 2(\alpha_2 P_2(t - \tau) S_2(t - \tau) - \gamma P_1(t - \tau) S_2) \beta_{21} (S_1 P_2 + S_2 P_1) + \gamma S_1(t - \tau) P_2^2 \tag{17}$$

Where $S_1 = \frac{\partial P_1}{\partial c_1}, S_2 = \frac{\partial P_2}{\partial c_1}$,

This sensitivity equation system (16) - (17) is then solved together with the original equation system (5)-(6) to approximate the sensitivity of the variable (P_1, P_2) to the parameter c_1 . The system variables' sensitivity to the function, and the related approach and statement apply.

Sensitivity of Variable to Parameter C_1

In Figure1, the variable consumption coefficient c_1 causes no difference and adjustment in the quality of the state vector allelochemical P_1 and P_2 which remain stable and tends to zero, as we decrease the value of $c_1 = 1.8$ to $c_1 = 2.6$. It indicates that the state will be less sensitive variable P_1 and P_2 to the variable c_1 . However, in the case of the equal rang of value of c_1 , the state variable amount of allelochemicals P_2 goes for the considerable change as shown by $\alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \gamma$. It indicates that when the delayed value of the coefficient decreases, the rate of allelochemicals increases. It also remains stable.

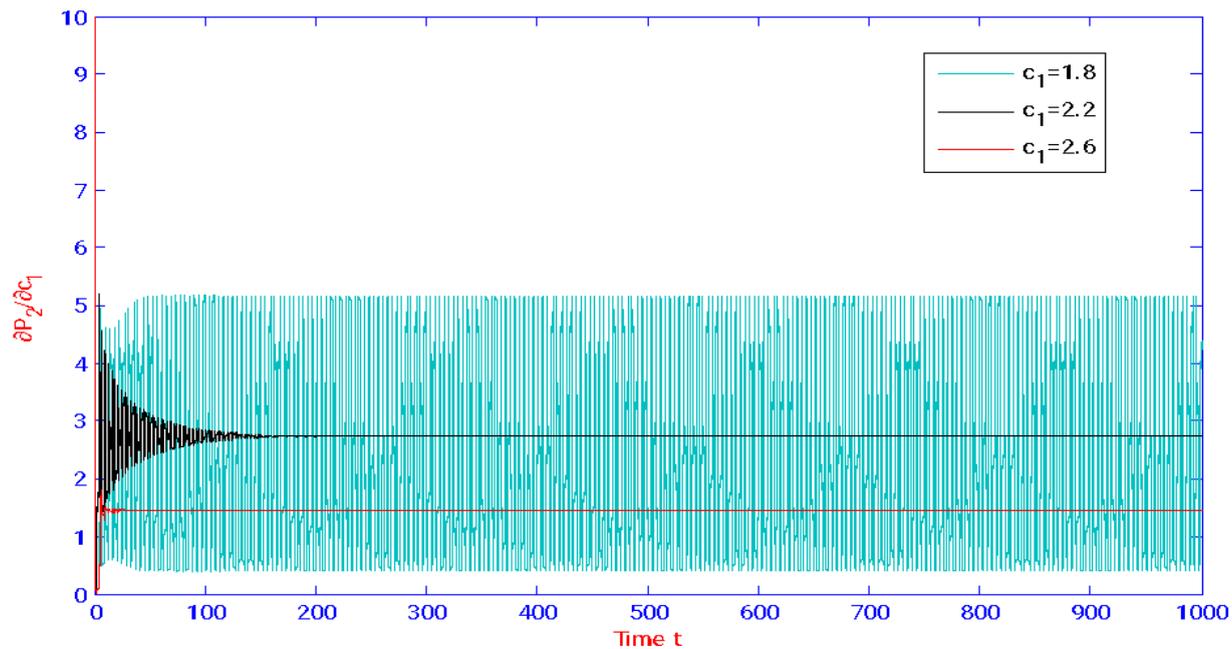


Figure 1: Time series graph between partial changes in allelochemicals P_1 for different value of coefficient c_1 .

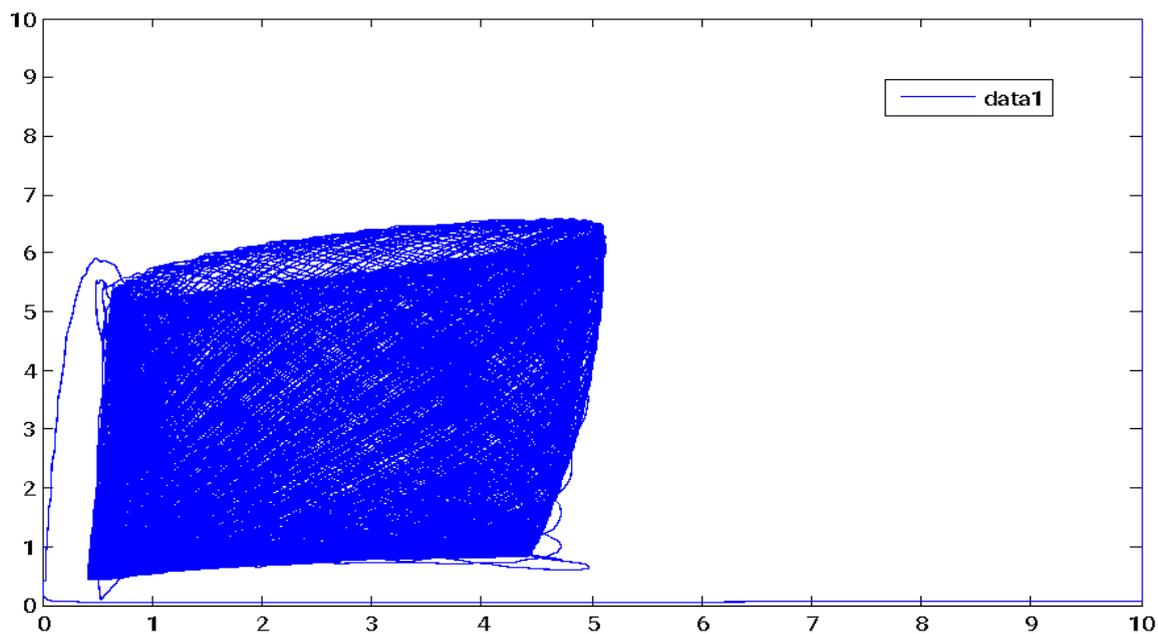


Figure 2: Phase diagram of plant populations P_1, P_2 when the value of $c_1 = 1.8$.

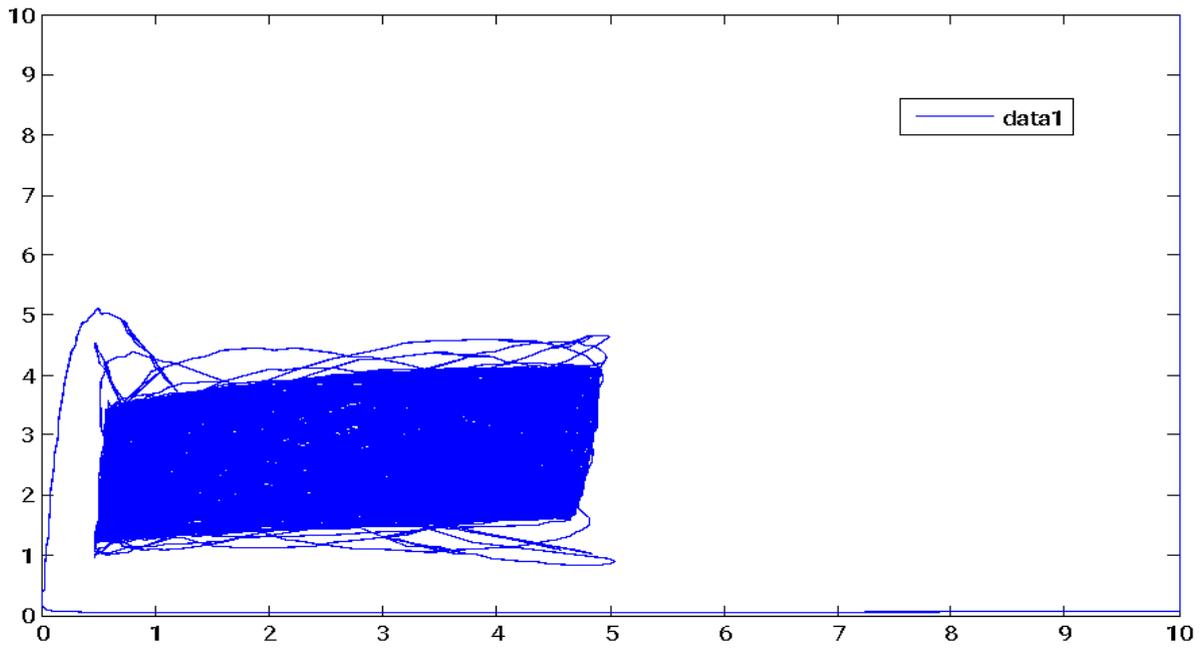


Figure 3: Phase diagram of plant populations P_1, P_2 when the value of $c_1 = 2.2$.

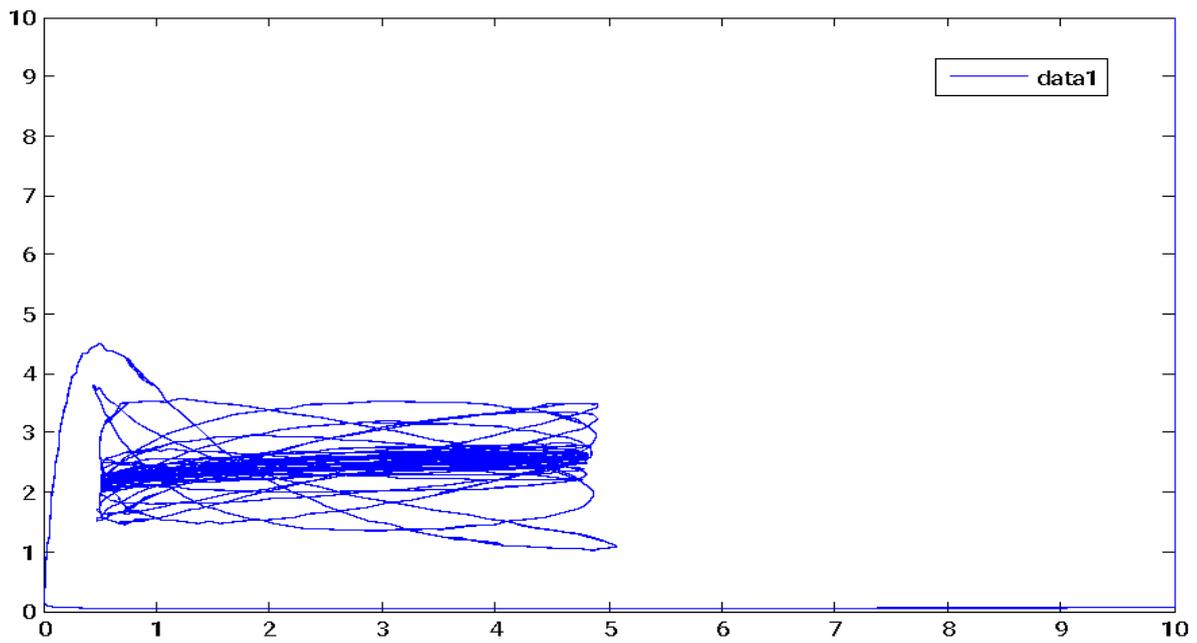


Figure 4: Phase diagram of plant populations P_1, P_2 when the value of $c_1 = 2.6$.

Numerical Example

The calculation is conducted with MATLAB to coordinate the analytical result with the aid of a numerical. The behavior of the system is demonstrated for the given sets of values:

$$c_1 = 1.8, c_2 = 2, \alpha_1 = 0.8, \alpha_2 = 0.7, \beta_{12} = 0.015, \beta_{21} = 0.05, \gamma = 0.008$$

The dynamical system (5)-(6) changes the behavior around the equilibrium, varying with delay parameter value.

III. RESULT AND DISCUSSION

Figure 1 represent the sensitivity of dynamical system with respect to the parameter c_1 . Which shows that the stability of the system oscillate with different value of $c_1 = 1.8$ to 2.6 , putting other parameters are constant. Kalra and Kumar studied the effect of toxic metal on sensitivity analysis of the dynamical system with parametric value $\beta = 0.3$ to 0.7 , $\gamma = 0.05$ to 0.2 and $\alpha = 0.5$ to 0.9 , and worked on the role of delay on two plant population with parametric value $k_N = 1$, $k_{NM} = 0.3$, $\alpha = 0.9$, $I = 0.5$, $\gamma = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.8$, $\delta_3 = 0.4$, found that in the absence of delay system is stable ($\tau = 0$) and at $\tau < 1.373$ system shows the asymptotically stable but when the value $\tau \geq 1.373$, system shows the Hopf-Bifurcation (3). Rihan's also conduct an experimental studied in point of subinterval $[0, 2\tau]$ on sensitivity analysis and observed that when change the value of τ in interval $[0, \tau]$ the system shows the change. Also observed that τ play a significant role in the field of sensitivity analysis (2).

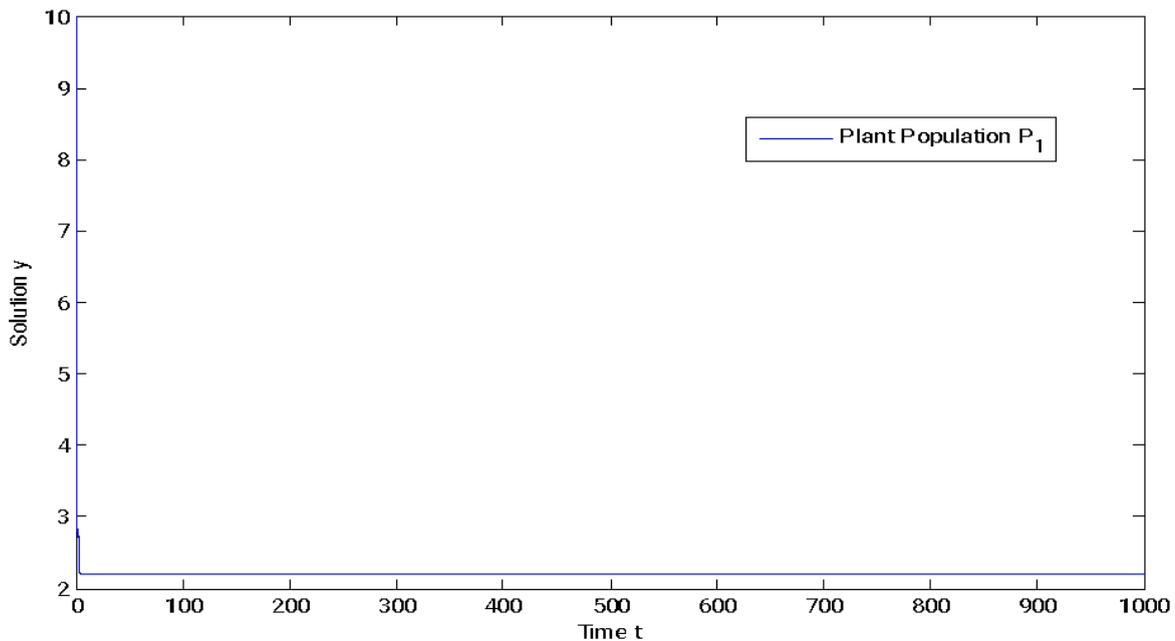


Figure 5: At $\tau = 0$, the plant population P_1 is stable.

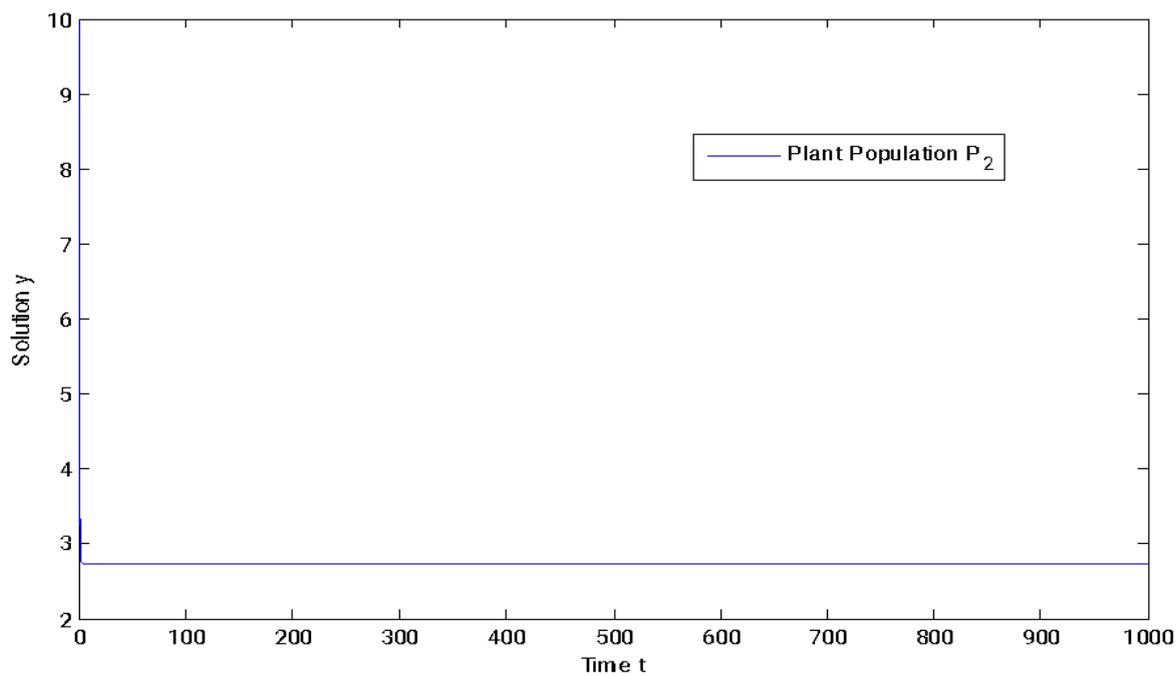


Figure 6: At $\tau = 0$, the plant population P_2 is stable.

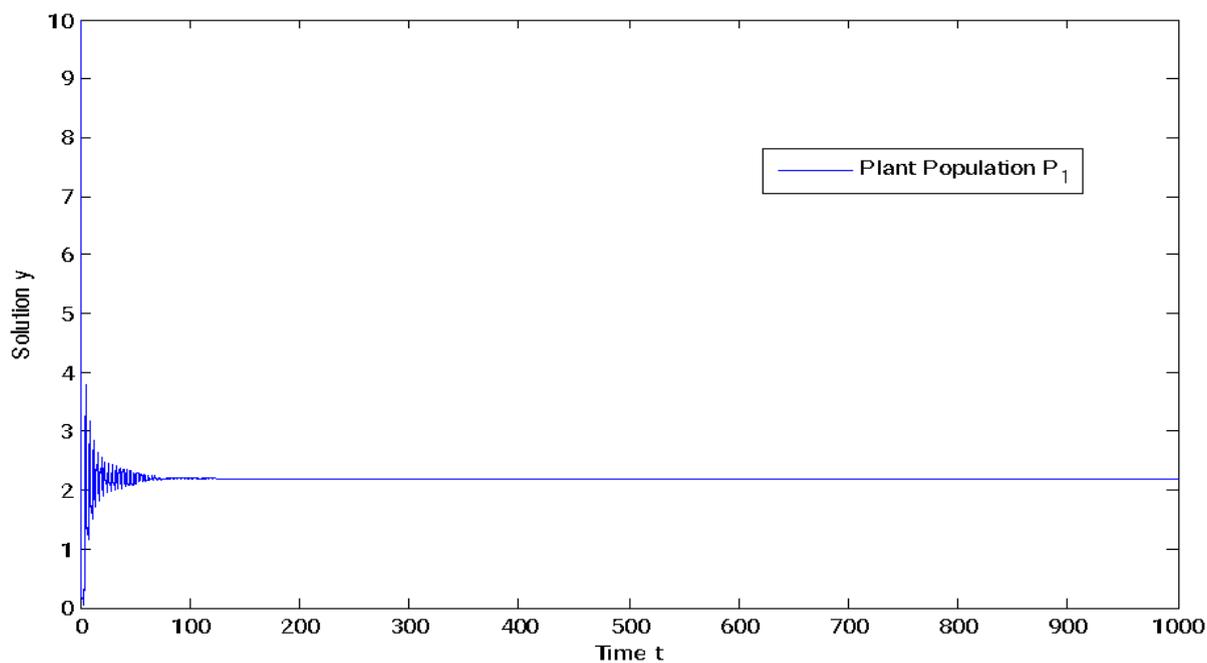


Figure 7: At $\tau < 0.8129$, the plant population P_1 loses its stability and shows the asymptotically stable.

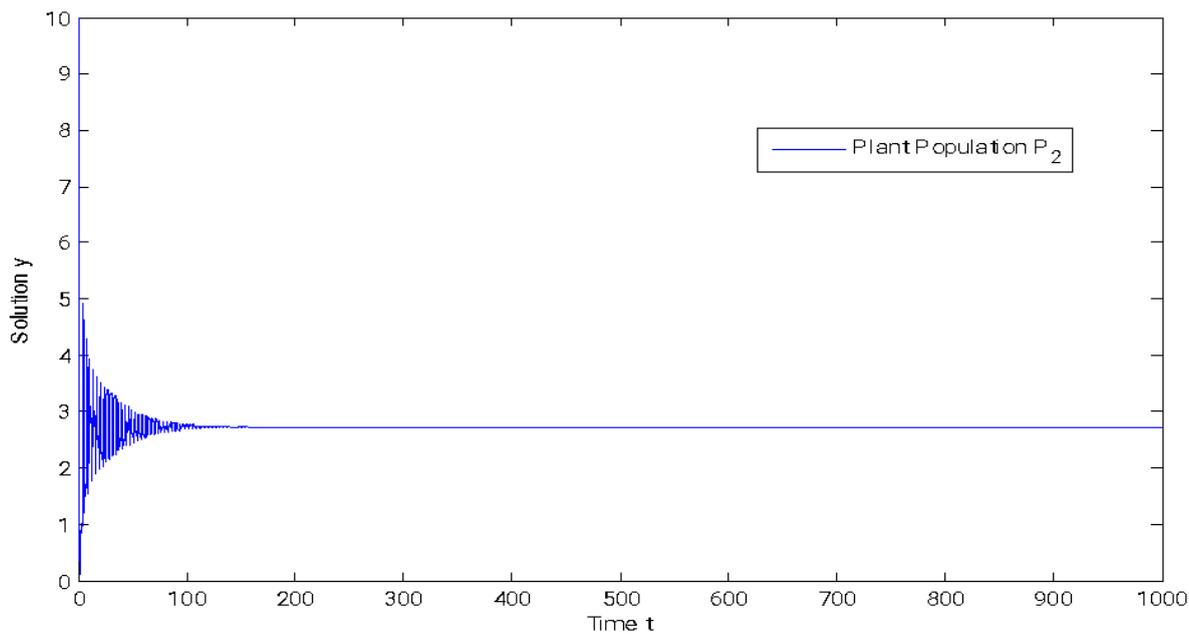


Figure 8: At $\tau < 0.8129$, the plant population P_2 loses its stability and shows the asymptotically stable.

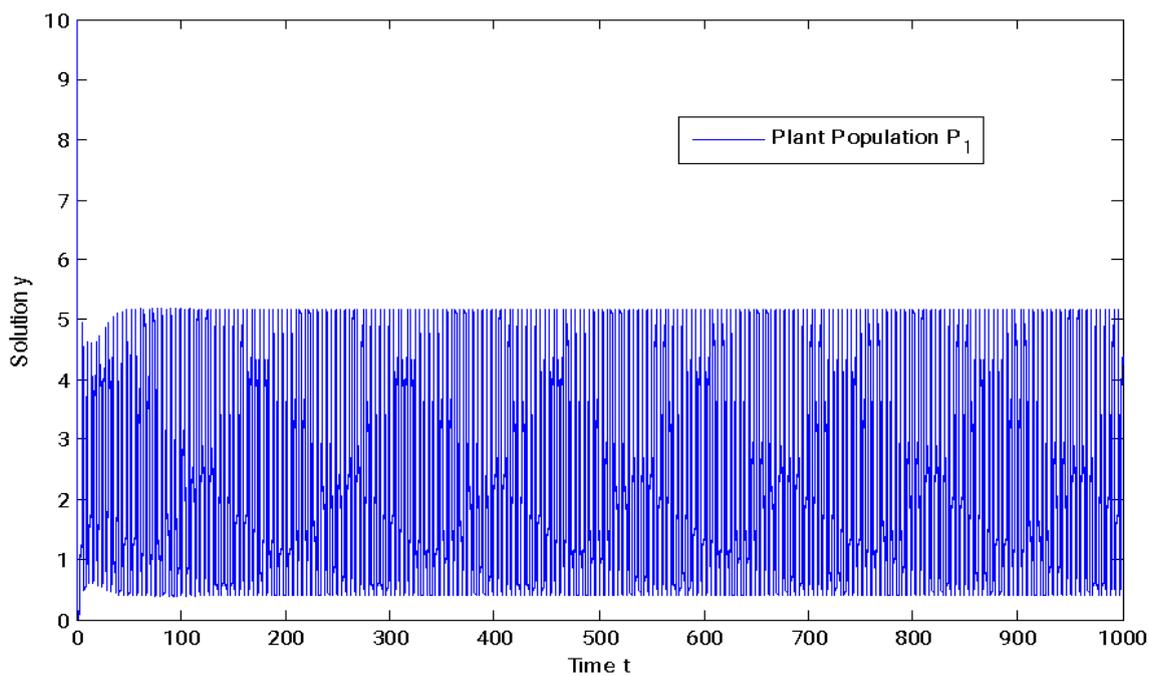


Figure 9: At $\tau \geq 0.9999$, the plant population P_1 loses its asymptotically stability and shows hopf-bifurcation.

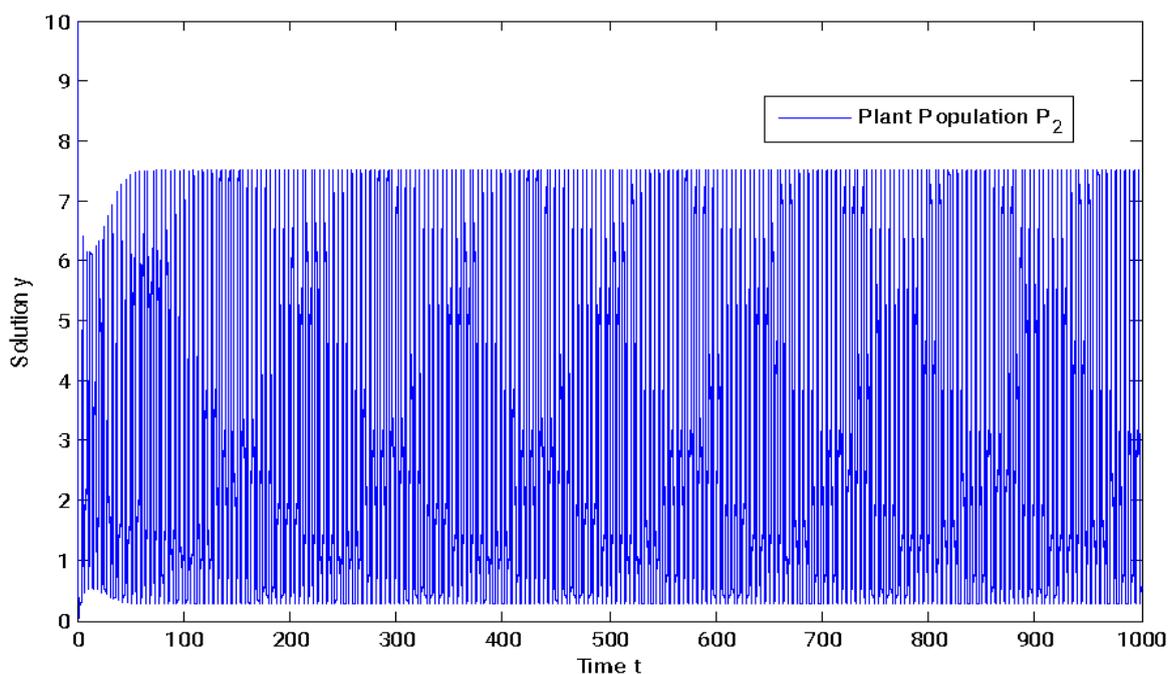


Figure 10: At $\tau \geq 0.9999$, the plant population P_2 loses its asymptotically stability and shows hopf-bifurcation.

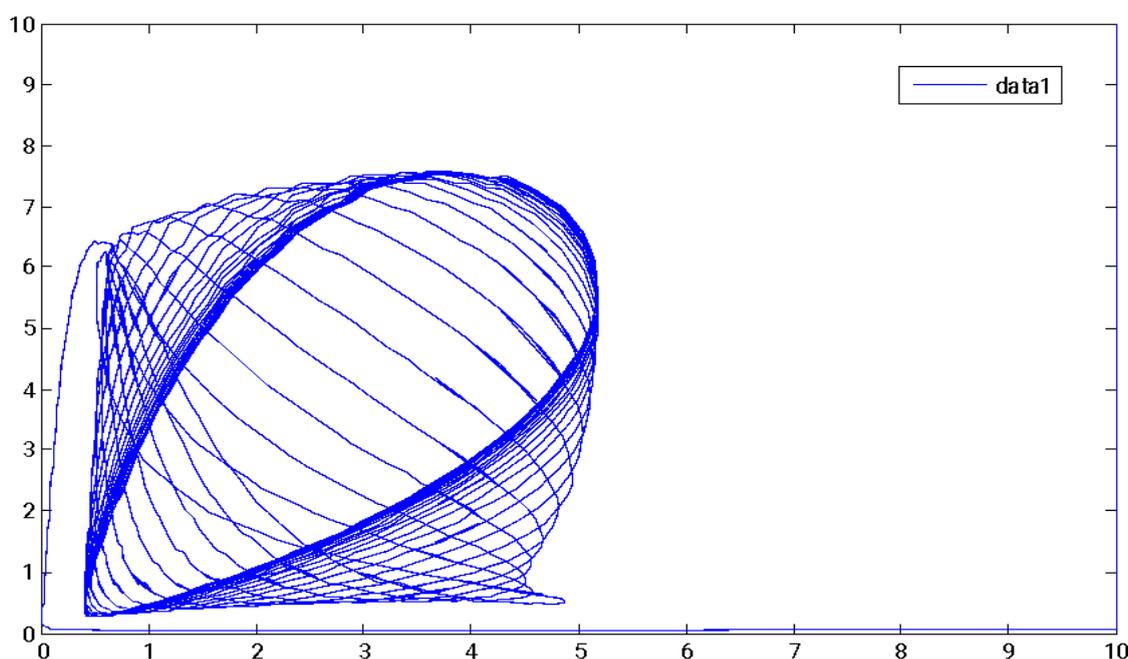


Figure 11: Phase diagram of the plant populations $\tau \geq 0.9999$.

IV. CONCLUSION

In this paper, we have studied the effect of allelochemicals on sensitivity and stability analysis of two competing plant populations. We have used the “Direct Method” to study the sensitivity analysis of the dynamics system. Further we have checked how the different parameters (cell proliferation, intraspecific and interspecific competition) affects the stability of the model. We have calculated the

sensitivity analysis for c_1 parameter by varying its value from $c_1 = 1.8$ to 2.2 to 2.6 . It has been observed that at $c_1 = 2.6$, stability in a dynamic system occurs. Further we have calculated the stability of the dynamic system about non-zero equilibrium point. It has been examined that when there is no delay, the system shows stability. And when the value of the delay parameter is less than the critical value, the system loses its stability and shows asymptotically stability. The equilibrium point loses its asymptotically stability when the value of the delay parameter is greater than the critical value and starts showing the Hopf-bifurcation. In MATLAB we use the `dde23` command for graphical representations.

Declarations

Ethical Approval

The declaration is “not applicable”.

Competing Interests

The declaration is “not applicable”.

Author's Contributions

Dipesh- Conceptualization, Formal analysis, Investigation, Methodology and Software

Pankaj Kumar- Supervision

Funding

The declaration is “not applicable”.

Availability of data and materials

The declaration is “not applicable”.

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