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ABSTRACT

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Keywords: pell equation, integral solutions, diophantine equations, pierpont primes, consecutive good prime, consecutive proth primes, pythagorean primes, brahma gupta lemma.

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ABSTRACT

Many researchers have been devoted to find the solutions (η, ζ) in the set of non-negative integers, of Diophantine equation (Pell Equation) of the type $\eta^2 = D\zeta^2 \pm \alpha$ where the value α is fixed positive integers. In this article, we look for non-trivial integer solutions to the negative Pell equation $x^2 = \phi y^2 - \psi^t$, where ϕ, ψ are the Pierpont Primes, Consecutive Good and Proth Primes, here $t \in N$, for the different choices of t particular by *(i) t = 1, (ii) t = 3, (iii) t = 5,(iv) t = 2k, (v) t = 2k + 5,for all* $k \in N$.

Keywords: pell equation, integral solutions, diophantine equations, pierpont primes, consecutive good prime, consecutive proth primes, pythagorean primes, brahma gupta lemma.

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I. INTRODUCTION

In this paper, Negative Pell equations are considered for their integral solutions in each of the three sections 4.1 to 4.3 as follows.

Pell's equation (also called Pell-Fermats equation) is any Diophantine equation $x \; - dy \; = 1$, where d is a given positive non-square integer and integer solutions are sought for x and y. In Cartesian coordinates, the equation has the form of a hyperbola; solutions occur whenever the curve passes through a point whose x and y coordinates are both integers, such as the trivial solution with x = 1 and y = 0 Joseph Loius proved that, as long as n is not a perfect square, Pell'sequation has infinitely many distinct integer solutions. These solutions may be used to accurately approximately the square root of n by rational number of the form $\frac{x}{y}$.

The negative Pell equation is given by $x^2 - dy^2 = -1$. It has also been extensively studied; it can be solved by the same method of continued fractions and will have solutions if and only if the period of the continued fraction has odd length. However it is not known which roots have been odd period lengths and therefore not known when the negative Pell equation is solvable. A necessary (but not sufficient) condition for solvability is that n is not divisible by 4 or by a prime of form $4k + 3$. Thus, for example, $x^2 - 3ny^2 = -1$ is never solvable, but $x^2 - 5ny^2 = -1$ may be.

Theorem 1.1. *If* (x_1, y_1) *is the fundamental solution of* $x^2 - dy^2 = 1$. *Then every positive solution of the equation is given by* (x_n, y_n), *where* x_n *and* y_n *are the integers determined from*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n, \; n = 1, 2, 3, \cdots$$

Proof. In anticipation of a contradiction, let us suppose that there exists a positive solution u, v that is not obtainable by the formula $(x_1 + y_1\sqrt{d})^n$. Because $x_1 + y_1\sqrt{d} > 1$, the powers of $x_1 + y_1\sqrt{d}$ become arbitrarily large; this means that $u + v\sqrt{d}$ must lie between two consecutive powers of $x_1 + y_1\sqrt{d}$, say.

$$(x_1 + y_1\sqrt{d})^n < u + v\sqrt{d} < (x_1 + y_1\sqrt{d})^{n+1}$$

or, to phrase it in different terms,

$$x_n + y_n\sqrt{d} < u + v\sqrt{d} < (x_n + y_n\sqrt{d})(x_1 + y_1\sqrt{d})$$

On multiplying this inequality by the positive number $x_n - y_n\sqrt{d}$ and nothing that $x_n^2 - dy_n^2 = 1$, we are led to

$$1 < (x_n - y_n\sqrt{d})(u + v\sqrt{d}) < x_1 + y_1\sqrt{d}$$

Next define the integers r and s by $r + s\sqrt{d} = (x_n - y_n\sqrt{d})(u + v\sqrt{d})$; that is, let

$$r = x_n u - y_n v d$$
$$s = x_n v - y_n u$$

An easy calculation reveals that

$$\begin{aligned} r^2 - ds^2 &= (x_n^2 - dy_n^2)(u^2 - dv^2) \\ &= 1 \end{aligned}$$

and therefore r, s is a solution of $x^2 - dy^2 = 1$ satisfying

$$1 < r + s\sqrt{d} < x_1 + y_1\sqrt{d}$$

Completion of the proof requires us to show that the pair (r, s) is a positive solution. Because $1 < r + s\sqrt{d}$ and $(r + s\sqrt{d})(r - s\sqrt{d}) = 1$, we find that $0 < r - s\sqrt{d} < 1$. In consequence

$$2r = (r + s\sqrt{d}) + (r - s\sqrt{d}) > 1 + 0 > 0$$

$$2s\sqrt{d} = (r + s\sqrt{d}) + (r - s\sqrt{d}) > 1 - 1 = 0$$

which makes both r and s positive. The upshot is that because, (x_1, y_1) is the fundamental solution of $x^2 - dy^2 = 1$, we must have $x_1 < r$ and $y_1 < s$; but then $x_1 + y_1\sqrt{d} < r + s\sqrt{d}$, violating an earlier inequality. This contradiction ends our argument.

Theorem 1.2. *Let p be a prime. The negative Pells equation $x^2 - py^2 = -1$ is solvable if and only if $p = 2$ or $p \equiv 1(mod4)$.*

Testing the solubility of the negative Pell equation:

Suppose D is a positive integer, not a perfect square. Then the negative Pell equation $x^2 - Dy^2 = -1$ is soluble if and only if D is expressible as $D = a^2 + b^2, \gcd(a, b) = 1$. a and b positive, b odd and the Diophantine equation $-bV^2 + 2aVW + bW^2 = 1$ has a solution(The Case of solubility occurs for exactly one such (a, b)).

The Algorithm:

(1) Find all expression of D a sum of two relatively prime squares using Cornacchia's method. If none, exist - the negative Pell equation is not soluble.
(2) For each representation $D = a^2 + b^2, \gcd(a, b) = 1$, a and b positive, b odd, test the solubility of $-bV^2 + 2aWV + bW^2 = 1$ using the Lagrange - Matthews algorithm. If solution exist - the negative Pell equation is soluble.
(3) If each representation yields no solution, then the negative Pell equation is insoluble.

II. SOLUTIONS OF PELL'S EQUATION INVOLVING PIERPONT PRIMES

In this section, concerns with the Pell equation $x^2 = 73y^2 - 3^t, t \in \mathbb{N}$, and infinitely many positive integer solutions are obtained for the choices of t given by $(i)t = 1,\ (ii)t = 3,\ (iii)t = 5, (iv)t = 2k$ and $(v)t = 2k + 5, k \in \mathbb{N}$.

A Pierpont prime is a prime number of the form $2^u 3^v + 1$ for some nonnegative integers u and v. Here using Pierpont primes 3 and 73 we form a Pell's equation $x^2 = 73y^2 - 3^t, t \in \mathbb{N}$ and search for its non-trivial integer solutions. A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are derived.

Choice 1: $t = 1$

The Pell equation is

$$x^2 = 73y^2 - 3 \tag{1}$$

Let (x_0, y_0) be the initial solution of (1) . Then $x_0 = 17;\ y_0 = 2$.
To find the other solutions of (1), consider the Pell equation

$$x^2 = 73y^2 + 1$$

whose initial solution $(\tilde{x_n}, \tilde{y_n})$ is given by

$$\tilde{x_n} = \frac{1}{2} f_n$$

$$\tilde{y_n} = \frac{1}{2\sqrt{73}} g_n$$

where

$$f_n = \frac{1}{2}[(2281249 + 267000\sqrt{73})^{n-1} + (2281249 - 267000\sqrt{73})^{n+1})]$$

$$g_n = \frac{1}{2\sqrt{73}}[(2281249 + 267000\sqrt{73})^{n-1} - (2281249 - 267000\sqrt{73})^{n+1})]$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions to (1) are obtained as

$$x_{n+1} = \frac{1}{2}[17f_n + 2\sqrt{73}g_n] \tag{2}$$

$$y_{n+1} = \frac{1}{2\sqrt{73}}[2\sqrt{73}f_n + 17g_n] \tag{3}$$

The recurrence relation satisfied by the solutions of (1) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$

$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

Choice 2: $t = 3$

The Pell equation is

$$x^2 = 73y^2 - 3^3 \tag{4}$$

with the initial solution $x_0 = 51;\ y_0 = 6$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[51f_n + 6\sqrt{73}g_n]$$
$$y_{n+1} = \frac{1}{2\sqrt{73}}[6\sqrt{73}f_n + 51g_n]$$

The recurrence relation satisfied by the solutions of (4) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

Choice 3: $t = 5$

The Pell equation is

$$x^2 = 73y^2 - 3^5 \tag{5}$$

with the initial solution $x_0 = 153;\ y_0 = 18$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[153f_n + 18\sqrt{73}g_n]$$
$$y_{n+1} = \frac{1}{2\sqrt{73}}[18\sqrt{73}f_n + 153g_n].$$

The recurrence relation satisfied by the solutions of (5) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

Choice 4: $t = 2k,\ \ k \in \mathbb{N}$

The Pell equation is

$$x^2 = 73y^2 - 3^{2k} \tag{6}$$

with the initial solution $x_0 = 1068(3^k);\ y_0 = 125(3^k)$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{3^k}{2}[1068f_n + 125\sqrt{73}g_n]$$
$$y_{n+1} = \frac{3^k}{2\sqrt{73}}[125\sqrt{73}f_n + 1068g_n]$$

The recurrence relation satisfied by the solutions of (6) are given by

$$x_{n+2} - 534000x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000y_{n+1} + y_n = 0$$

Choice 5: $t = 2k + 5, k \in \mathbb{N}$

The Pell equation is

$$x^2 = 73y^2 - 3^{2k+5} \tag{7}$$

with the initial solution $x_0 = 5861(3^{k-1});\ y_0 = 686(3^{k-1})$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{3^{k-1}}{2}[5861 f_n + 686\sqrt{73} g_n]$$
$$y_{n+1} = \frac{3^{k-1}}{2\sqrt{73}}[686\sqrt{73} f_n + 5861 g_n]$$

The recurrence relations satisfied by the solutions of (7) are given by

$$x_{n+2} - 534000 x_{n+1} + x_n = 0$$
$$y_{n+2} - 534000 y_{n+1} + y_n = 0$$

III. INTEGRAL SOLUTIONS OF NEGATIVE PELL'S EQUATION INVOLVING CONSECUTIVE GOOD Primes $x^2 = 41y^2 - 37^t$

Let $d \neq 1$ be a positive non - square integer and N be any fixed positive integer. Then the equation $x^2 - dy^2 = \pm N$ is known as Pell's equation named after the famous Mathematician John Pell.

A *good prime* is a prime number whose square is greater than the product of any two primes at the same number of positions before and after it in the sequence of primes.
That is, a good prime satisfies the inequality $p_n^2 > p_{n-i} \times p_{n+i}$, for all $1 \leq i \leq n-1$ where p_n is the n^{th} prime.

In this section, we fix d and N to be two consecutive good primes 41 and 37 and search for non - trivial integer solution to the equation $x^2 = 41y^2 - 37^t, t \in \mathbb{N}$ for the different choices of t given by $(i) t = 1,\ (ii) t = 3,\ (iii) t = 5,\ (iv) t = 2k,\ \forall k \in \mathbb{N}$ and $(v) t = 2k + 5,\ \forall k \in \mathbb{N}$. Further, recurrence relation on the solutions are obtained.

By testing the solubility of the negative Pell equation, solving $x^2 + y^2 = 41$ we get $(x, y) = (5, 4)$. Number of positive primitive solutions with $x \geq y$ is 1.

(1): Testing $(a, b) = (4, 5)$.
(2): $-bV^2 + 2aVW + bW^2 = 1$ has a solution $(V, W) = (2, 1)$.
So $x^2 - 41y^2 = -1$ is solvable.

Choice 1: $t = 1$
The Pell equation is

$$x^2 = 41y^2 - 37 \tag{8}$$

with the initial solution $x_0 = 2;\ y_0 = 1$.

To find the other solutions, consider the more general Pell equation $x^2 = 41y^2 + 1$ whose general solution $(\tilde{x_n}, \tilde{y_n})$ is given by

$$\tilde{x_n} = \frac{1}{2} f_n;\ \tilde{y_n} = \frac{1}{2\sqrt{41}} g_n.$$

where $f_n = (2049 + 320\sqrt{41})^{n+1} + (2049 - 320\sqrt{41})^{n+1}$ and $g_n = (2049 + 320\sqrt{41})^{n+1} - (2049 - 320\sqrt{41})^{n+1},\ n = 0, 1, \cdots$.
Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[2 f_n + \sqrt{41} g_n]$$
$$y_{n+1} = \frac{1}{2\sqrt{41}}[\sqrt{41} f_n + 2 g_n]$$

The recurrence relations satisfied by the solutions of (8) are given by

$$x_{n+2} - 4098 x_{n+1} + x_n = 0$$
$$y_{n+2} - 4098 y_{n+1} + y_n = 0$$

Choice 2: $t = 3$
The Pell equation is

$$x^2 = 41y^2 - 37^3 \tag{9}$$

with the initial solution $x_0 = 254;\ y_0 = 53$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as $x_{n+1} = \frac{1}{2}[254f_n + 53\sqrt{41}g_n]$ and $y_{n+1} = \frac{1}{2\sqrt{41}}[53\sqrt{41}f_n + 254g_n]$.

The recurrence relations satisfied by the solutions of (9) are given by

$$x_{n+2} - 4098x_{n+1} + x_n = 0$$
$$y_{n+2} - 4098y_{n+1} + y_n = 0$$

Choice 3: $t = 5$
The Pell equation is

$$x^2 = 41y^2 - 37^5 \tag{10}$$

with the initial solution $x_0 = 9398;\ y_0 = 1961$.
Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[9398f_n + 1961\sqrt{41}g_n]$$
$$y_{n+1} = \frac{1}{2\sqrt{41}}[1961\sqrt{41}f_n + 9398g_n]$$

The recurrence relations satisfied by the solutions of (10) are given by

$$x_{n+2} - 4098x_{n+1} + x_n = 0$$
$$y_{n+2} - 4098y_{n+1} + y_n = 0$$

Choice 4: $t = 2k,\ k \in \mathbb{N}$

The Pell equation is

$$x^2 = 41y^2 - 37^{2k} \tag{11}$$

with the initial solution $x_0 = 32(37^k);\ y_0 = 5(37^k)$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non - zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{37^k}{2}[32f_n + 5\sqrt{41}g_n]$$
$$y_{n+1} = \frac{37^k}{2\sqrt{41}}[5\sqrt{41}f_n + 32g_n].$$

The recurrence relations satisfied by the solutions of (11) are given by

$$x_{n+2} - 4098x_{n+1} + x_n = 0$$
$$y_{n+2} - 4098y_{n+1} + y_n = 0$$

Choice 5: $t = 2k + 5, k \in \mathbb{N}$
The Pell equation is

$$x^2 = 41y^2 - 37^{2k+5} \tag{12}$$

with the initial solution $x_0 = 347726(37^{k-1});\ y_0 = 72557(37^{k-1})$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{37^{k-1}}{2}[347726 f_n + 72557\sqrt{41} g_n]$$

$$y_{n+1} = \frac{37^{k-1}}{2\sqrt{41}}[72557\sqrt{41} f_n + 347726 g_n]$$

The recurrence relations satisfied by the solutions of (12) are given by

$$x_{n+2} - 4098x_{n+1} + x_n = 0$$
$$y_{n+2} - 4098y_{n+1} + y_n = 0$$

IV. OBSERVATION ON NEGATIVE PELL'S EQUATION INVOLVING CONSECUTIVE PROTH PRIMES $x^2 = 13y^2 - 17^t$

In this section, we fix d and N to be two consecutive proth primes 13 and 17 and search for non - trivial integer solution to the equation $x^2 = 13y^2 - 17^t, t \in \mathbb{N}$ for the different choices of t given by $(i) t = 1,\ (ii) t = 3,\ (iii) t = 5,\ (iv) t = 2k,\ \forall k \in \mathbb{N}$ and $(v) t = 2k + 5,\ \forall k \in \mathbb{N}$. Further, recurrence relation on the solutions are obtained.

By testing the solubility of the negative Pell equation, solving $x^2 + y^2 = 13$ we have $(x, y) = (3, 2)$. Number of positive primitive solutions with $x \geq y$ is 1.

(1): Testing $(a, b) = (2, 3)$.
(2): $-bV^2 + 2aVW + bW^2 = 1$ has a solution $(V, W) = (71, 38)$.
So $x^2 - 13y^2 = -1$ is solvable.

Choice 1: $t = 1$

The Pell equation is

$$x^2 = 13y^2 - 17 \tag{13}$$

with the initial solution $x_0 = 10;\ y_0 = 3$.

To find the other solutions, consider the more general Pell equation $x^2 = 13y^2 + 1$ whose general solution $(\tilde{x_n}, \tilde{y_n})$ is given by

$$\tilde{x_n} = \frac{1}{2} f_n;\ \tilde{y_n} = \frac{1}{2\sqrt{13}} g_n.$$

where $f_n = (649 + 180\sqrt{13})^{n+1} + (649 - 180\sqrt{13})^{n+1}$ and $g_n = (649 + 180\sqrt{13})^{n+1} - (649 - 180\sqrt{13})^{n+1},\ n = 0, 1, \cdots$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[10 f_n + 3\sqrt{13} g_n]$$

$$y_{n+1} = \frac{1}{2\sqrt{13}}[3\sqrt{13} f_n + 10 g_n]$$

The recurrence relations satisfied by the solutions of (13) are given by

$$x_{n+2} - 1298x_{n+1} + x_n = 0$$
$$y_{n+2} - 1298y_{n+1} + y_n = 0$$

Choice 2: $t = 3$
The Pell equation is

$$x^2 = 13y^2 - 17^3 \quad (14)$$

with the initial solution $x_0 = 350;\ y_0 = 99$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as $x_{n+1} = \frac{1}{2}[350f_n + 99\sqrt{13}g_n]$ and $y_{n+1} = \frac{1}{2\sqrt{13}}[99\sqrt{13}f_n + 350g_n]$.

The recurrence relations satisfied by the solutions of (14) are given by

$$x_{n+2} - 1298x_{n+1} + x_n = 0$$
$$y_{n+2} - 1298y_{n+1} + y_n = 0$$

Choice 3: $t = 5$
The Pell equation is

$$x^2 = 13y^2 - 17^5 \quad (15)$$

with the initial solution $x_0 = 1270;\ y_0 = 483$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[1270f_n + 483\sqrt{13}g_n]$$
$$y_{n+1} = \frac{1}{2\sqrt{13}}[483\sqrt{13}f_n + 1270g_n]$$

The recurrence relations satisfied by the solutions of (15) are given by

$$x_{n+2} - 1298x_{n+1} + x_n = 0$$
$$y_{n+2} - 1298y_{n+1} + y_n = 0$$

Choice 4: $t = 2k,\ k \in \mathbb{N}$
The Pell equation is

$$x^2 = 13y^2 - 17^{2k} \quad (16)$$

with the initial solution $x_0 = 18(17^k);\ y_0 = 5(17^k)$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non - zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{17^k}{2}[18f_n + 5\sqrt{13}g_n]$$
$$y_{n+1} = \frac{17^k}{2\sqrt{13}}[5\sqrt{13}f_n + 18g_n].$$

The recurrence relations satisfied by the solutions of (16) are given by

$$x_{n+2} - 1298x_{n+1} + x_n = 0$$
$$y_{n+2} - 1298y_{n+1} + y_n = 0$$

Choice 5: $t = 2k + 5, k \in \mathbb{N}$

The Pell equation is

$$x^2 = 13y^2 - 17^{2k+5} \tag{17}$$

with the initial solution $x_0 = 203570(17^{k-1});\ y_0 = 56739(17^{k-1})$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$, the sequence of non- zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{17^{k-1}}{2}[203570f_n + 56739\sqrt{13}g_n]$$
$$y_{n+1} = \frac{17^{k-1}}{2\sqrt{13}}[56739\sqrt{13}f_n + 203570g_n]$$

The recurrence relations satisfied by the solutions of (17) are given by

$$x_{n+2} - 1298x_{n+1} + x_n = 0$$
$$y_{n+2} - 1298y_{n+1} + y_n = 0$$

V. CONCLUSION

Solving a Pells equation using the above method provides powerful tool for finding solutions of equations of similar type. Neglecting any time consideration it is possible using current methods to determine the solvability of Pell like equation.

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