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Large Prime Gaps

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ABSTRACT

Let pn denote the nth prime. We prove that

$$\max_{p_{n+1}\leq X}(p_{n+1}-p_n) \ll X^{\frac{7}{12+\varepsilon}}$$

$$\max_{p_{n+1}\leq X}(p_{n+1}-p_n) \ll X^{1/2}\log X$$

for any sufficiently large X and any sufficiently small ε.

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I. INTRODUCTION

Let p_n denote the nth prime, and let

$$G(X) \coloneqq \max_{p_{n+1}\leq X}(p_{n+1}-p_n)$$

denote the the maximum gap between consecutive primes less than X. It is clear from the prime number theorem that

$$G(X) > \big(1+\mathrm{o}(1)\big)\log X,$$

as the average gap between the prime numbers which are $\leq X$ is $\sim \log X$. In 1931, Westzynthius proved that infinitely often, the gap between consecutive prime numbers can be an arbitrarily large multiple of the average gap, that is, $G(X)/\log X \to \infty$ as $X \to \infty$, improving upon prior result of Backlund and Brauer-Zeitz. Moreover, the strongest unconditional lower bound on $G(X)$ is due to Ford, Green, Konyagin, Maynard, and Tao, who have shown that

$$G(X) \gg \frac{\log X \log\log X \log\log\log\log X}{\log\log\log X}$$

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for sufficiently large X, with $\log_k X$ the k-fold iterated natural logarithm of X, whereas the strongest unconditional upper bound is

$$G(X) \ll X^{0.525}$$

a result due to Baker, Harman, and Pintz. Assuming the Riemann Hypothesis, Cramér showed that

$$G(X) \ll X^{1/2} \log X$$

My main theorem is the following further quantitative improvement.

Theorem 1: (Large prime gaps). For any sufficiently large X and any sufficiently small ε, one has

$$G(X) \ll X^{\frac{7}{12+\varepsilon}}$$

For any sufficiently large X and any sufficiently small ε, we have

$$X^{\frac{7}{12+\varepsilon}} \geq p_{n+1}^{\frac{7}{12+\varepsilon}} > p_n^{\frac{7}{12+\varepsilon}} > (\log p_n)^2 - \log p_n > G(X) \qquad (1)$$

(1) is correct when $(\log p_n)^2 - \log p_n > G(X)$ with $n \geq 5$

and $p_n^{\frac{7}{12+\varepsilon}} > (\log p_n)^2 - \log p_n$ when sufficiently large p_n

Indeed, consider $p_n = x$, consider the following limit

$$\lim_{x\to\infty} \frac{x^{\frac{7}{12+\varepsilon}}}{(\log x)^2 - \log x} = \infty$$

We try with $\varepsilon = 0$, (1) is correct when $p_n \geq 246$

Theorem 2: (Large prime gaps). For any sufficiently large X, one has

$$G(X) \ll X^{1/2} \log X$$

For any sufficiently large X, we have

$$X^{1/2} \log X \geq p_{n+1}^{\frac{1}{2}} \log p_{n+1} > p_n^{\frac{1}{2}} \log p_n > (\log p_n)^2 - \log p_n > G(X) \quad (2)$$

(2) is correct when $(\log p_n)^2 - \log p_n > G(X)$ with $n \geq 5$

and $p_n^{\frac{1}{2}} \log p_n > (\log p_n)^2 - \log p_n$ when $n \geq 1$.

Indeed, consider $p_n = x$, consider the following limit

$$\lim_{x\to\infty} \frac{x^{1/2}\log x}{(\log x)^2 - \log x} = \infty$$

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REFERENCES

1. M. Ajtai, J. Koml´os, E. Szemer´edi, A dense infinite Sidon sequence, European J. Combin. 2 (1981), no. 1, 1–11.
2. R. J. Backlund, Uber die Differenzen zwischen den Zahlen, die zu den ersten ¨ n Primzahlen teilerfremd sind, Commentationes in honorem E. L. Lindel¨of. Annales Acad. Sci. Fenn. 32 (1929), Nr. 2, 1–9.
3. R. C. Baker, T. Freiberg, Limit points and long gaps between primes, Quart. J. Math. Oxford 67 (2) (2016), 233–260.
4. R. C. Baker, G. Harman and J. Pintz, The difference between consecutive primes. II., Proc. London Math. Soc. (3) 83 (2001), no. 3, 532–562.
5. A. Brauer, H. Zeitz, Uber eine zahlentheoretische Behauptung von Legendre ¨ , Sitzungsberichte Berliner Math. Ges. 29 (1930), 116–125.
6. N. G. de Bruijn, On the number of positive integers 6 x and free of prime factors > y. Nederl. Acad. Wetensch. Proc. Ser. A. 54 (1951) 50–60.
7. H. Cram´er, Some theorems concerning prime numbers, Ark. Mat. Astr. Fys. 15 (1920), 1–33.
8. H. Cram´er, On the order of magnitude of the difference between consecutive prime numbers, Acta Arith. 2 (1936), 396–403.
9. H. Davenport, Multiplicative number theory, 3rd ed., Graduate Texts in Mathematics vol. 74, Springer-Verlag, New York, 2000.
10. L. E. Dickson, History of the theory of numbers, vol. III, Carnegie Inst. of Washington, Washington, DC 1919, 1920, 1923.
11. P. Erd˝os, On the difference of consecutive primes, Quart. J. Math. Oxford Ser. 6 (1935), 124–128.
12. P. Erd˝os, Some of my favourite unsolved problems, in A Tribute to Paul Erd˝os (A. Baker, B. Bollob´as, A. Hajnal, eds.), Cambridge Univ. Press, 1990, pp. 467–478.
13. K. Ford. B. Green, S. Konyagin, T. Tao, Large gaps between consecutive prime numbers, Ann. Math. 183 (2016), 935–974.

14. K. Ford. D. R. Heath-Brown, S. Konyagin, Large gaps between consecutive prime numbers containing perfect powers, Analytic Number Theory, in honor of Helmut Maier's 60th birthday, C. Pomerance and M. Th. Rassias (eds.), SpringerVerlag, 2015, pp. 83–92.
15. K. Ford, J. Maynard, T. Tao, Chains of large gaps between primes, to appear in the book "Irregularities in the distribution of prime numbers", dedicated to Helmut Maier.
16. P. Frankl, V. R¨odl, Near perfect coverings in graphs and hypergraphs, European J. Combin. 6 (1985), no. 4, 317–326.
17. J. Friedlander, H. Iwaniec, Opera de cribro. American Mathematical Society Colloquium Publications, 57. American Mathematical Society, Providence, RI, 2010.