On Sufficient Conditions for Absolute Convergence of Fourier Series of Almost-Periodic Bezikovich Functions

Farkhodzhon Makhmadshevich Talbakov
Tajik State Pedagogical University

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Author: Tajik State Pedagogical University named after S.Aenya, 121 Rudaki str, Dushanbe 734003, Tajikistan.

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I. INTRODUCTION

Let $B_p \quad (1 \leq p \leq \infty)$ be a linear space consisting of measurable functions $f(x)$ for which $|f(x)|^p \quad (1 \leq p < \infty)$ is Lebesgue integrable on any finite segment of the real axis with norm

$$||f||_{B_p} = \{\overline{M}[|f(x)|^p]\}^{\frac{1}{p}} = \{\overline{\lim_{T\to\infty}} \int_{-T}^{T} |f(x)|^p dx\}^{\frac{1}{p}} < \infty,$$

$$||f||_{B_\infty} = vrai \sup_{-\infty<x<\infty} |f(x)| < \infty.$$

At $1 \leq p < \infty$, A. Bezikovich [1] or [2], introduced the following concept of B_p–almost–periodic function.

Definition 1. A function $f(x)$ is called almost-periodic in the sense of Bezikovich or B_p-almost-periodic if there exists a sequence of finite trigonometric polynomials $\{P_n(x)\}$ of the form

$$P_n(x) = \sum_{k=1}^{n} A_k(f) e^{i\lambda_k x},$$

for which the following condition holds

$$\lim_{n\to\infty} ||f(x) - P_n(x)||_{B_p} = 0.$$

For each $f \in B_p$, a function is defined

$$a(f,\lambda) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} f(x)e^{-i\lambda x}dx = M\{f(x)e^{-i\lambda x}\}.$$

It can differ from zero no more than on the countable set of values of λ: λ_1, λ_2,..., λ_k,... . The numbers $\{\lambda_k\}$ are called exponents Fourier transform or the spectrum of the function in question, and the numbers $A_k(f) = a(f,\lambda_k)$ - Fourier coefficients. Thus, each function $f \in B_p$ can be written a Fourier series

$$f(x) \sim \sum_k A_k(f)e^{i\lambda_k x}.$$

The space B_∞ of uniform almost-periodic functions denote B (see, for example, [3], [4]).

Denote by $\Delta_t^m f(x)$ the finite difference of the mth order of the function $f \in B_p$ at the point x with a step of t, i.e.

$$\Delta_t^m f(x) = \sum_{r=0}^{m} (-1)^{m-r} \tbinom{m}{r} f(x+rt).$$

To determine the smoothness of the function, the quality of the structural characteristic of the function $f \in B_p$, $p \geq 1$, we will use the continuity module of the order k

$$\omega_m(f,h)_{B_p} = \sup_{|t|\leq h} ||\Delta_t^m f(x)||_{B_p}, \quad h > 0, m \in N.$$

Let Π be any partition $-T = x_0 < x_1 < x_2 < ... < x_n = T$ of the function $f(x)$ of interval $(-T,T)$. Given $r \geq 1$, $T > 0$ and a positive m, we write

$$V_{r,T}^m(f) = [\sup \sum_{r=0}^{n-1} |\Delta_{h_r}^m f(x_r)|^r]^{1/r},$$

where $m \in N$, $h_r = \frac{x_{r+1}-x_r}{m}$. Then we call the value

$$V_r^m(f) = \overline{\lim_{T\to\infty}} \frac{1}{2T} V_{r,T}^m(f)$$

r-the variation of function of order m.

In [9]-[17] and others, some necessary and sufficient conditions for the absolute convergence of the Fourier series of almost-periodic functions in the sense of Bohr and Bezikovich were obtained.

J. Museliak [10] showed that if the spectrum $\lambda_k \to \infty$ and $k^\alpha = O(\lambda_k)$, $k \to \infty$, $\alpha > 0$, then for the function $f \in B_2$ the condition

$$\sum_{k=1}^{\infty} k^{\frac{1-\frac{\beta}{2}}{\alpha-1}} \omega_1^{\beta}(f; \frac{1}{k})_{B_2} < \infty, \tag{1}$$

at $0 < \beta < 2$, entails the convergence of the series

$$\sum_{n=1}^{\infty} |A_k(f)|^{\beta}. \tag{2}$$

N.P. Kuptsov [11] showed that for functions $f \in B$, condition (1) for $\alpha = 1$, $\beta = 1$ and replacing the value $\omega_1(f, \frac{1}{k})_{B_2}$ by $\omega_2(f, \frac{1}{k})_B$ provides absolute convergence of the series (2).

In the work of A.G. Pritula [12] it is proved that if for $\lambda_k \to \infty$, $0 < \beta < q$, $2 \leq q < \infty$ $(\frac{1}{p} + \frac{1}{q} = 1)$, $\gamma > 0$ the condition is met

$$\sum_{\nu=1}^{\infty} (\frac{\lambda_{2^\nu}}{\lambda_{2^{\nu-1}}})^{\beta} \omega_1^{\beta}(f, \frac{1}{\lambda_{2^\nu}})_{B_p} 2^{\nu(\gamma + \frac{q-\beta}{q})} < \infty,$$

that

$$\sum_{k=1}^{\infty} |A_k|^{\beta} k^{\beta} < \infty.$$

In the case when $f(x) \in B_p$, $1 < p \leq 2$, $\lambda_k \to 0$, A. S. Jafarova and G. A. Mammadova [13] established the convergence of the series

$$\sum_{k=1}^{\infty} |A_k|^{\beta} \varphi(k),$$

with some restrictions on the functions $\varphi(k)$. Instead of the continuity modulus, they used the following value based on the Laplace transform

$$\Omega(f; H; \delta; \theta) = \delta \sup_x |\int_0^{\infty} exp(-\delta\theta) f(x-t) exp(i\theta t) dt|, \quad \delta > 0, \theta \in R.$$

In the work of Yu.K. Khasanov [14], some sufficient, and in the case of monotonous decrease of the Fourier coefficients, the necessary conditions for the absolute convergence of the Fourier series of almost-periodic Bezikovich functions are established when the Fourier exponents have a single limiting point at infinity or at zero.

The results of this note are analogs of some results of [10], [14] and [15] for the class of almost-periodic Bezikovich functions.

0.1 Main results

1.1. The note discusses some new sufficient conditions for the absolute convergence of Fourier series of almost-periodic functions from the space B_2 when the spectrum $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ has a single limit point at infinity, i.e.

$$\lambda_0 = 0; \quad \lambda_{-k} = -\lambda_k; \quad |\lambda_k| < |\lambda_{k+1}|; \quad \lim_{k\to\infty} \lambda_k = \infty.$$

It is well known that for an arbitrary function $f \in B_2$ having a Fourier series expansion

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k(f)\cos\lambda_k x + b_k(f)\sin\lambda_k)x, \tag{3}$$

где

$$a_0(f) = M\{f(x)\},$$

$$a_k(f) = M\{f(x)\cos\lambda_k x\},$$

$$b_k(f) = M\{f(x)\sin\lambda_k x\} \quad (k = 1, 2, ...),$$

$$M\{g(x)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(x)dx.$$

We prove the following theorem concerning the absolute convergence of series (4) with known coefficients (see, for example, [3], [10-15]).

Theorem 1. *Let $f(x) \in b_2$ is a limited function. Suppose that the function $\Phi(u)$ is non-decreasing such that $\Phi(u) > 0$ if $u > 0$ and $u^2/(\Phi(u))$ is also a non-decreasing function. If at $0 < \beta < 1$ is executed*

$$\sum_{\nu=1}^{\infty}[\mu(2^{\nu}\pi) - \mu(2^{\nu-1}\pi) + 1]^{1-\frac{\beta}{2}}\omega^{\beta}(f, 2^{-\nu})\omega_{\Phi}^{\beta/2}(f, 2^{-\nu})\Phi^{-\beta/2}[\omega(f, 2^{-\nu})] < \infty, \tag{4}$$

where

$$\omega(f, h) = vrai\sup_{x}\sup_{|\delta|\le h}|f(x+\delta) - f(x)|,$$

$$\omega_{\Phi}(f, h) = \sup_{|\delta|\le h}\overline{M}\{|f(x+\delta) - f(x)|\},$$

that row

$$\sum_{k=1}^{\infty}(|a_k(f)|^{\beta} + |b_k(f)|^{\beta}) \tag{5}$$

it fits.

Proof We first prove an important inequality

$$\sum_{n\in A_\nu}(|a_k(f)|^2+b_k(f)|^2)\le\frac{1}{2}M\{|f(x+2^{-\nu-1})-f(x-2^{-\nu-1})|^2\}, \tag{6}$$

where $A_\nu=E_k\{2^{-\nu-1}\pi\le\lambda_k\le 2^{-\nu}\pi\}$, $\nu\ge 1$.

For any $h\ge 0$, consider the function

$$F_h(x)=f(x+h)-f(x-h).$$

The coefficients of the Fourier function $F_h(x)$ are defined as follows:

$$a_0(F_h)=M\{F_h(x)\}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[f(x+h)-f(x-h)]dx=$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[f(x)-f(x)]dx=0.$$

$$a_k(F_h)=M\{F_h(x)\cos\lambda_k x\}=M\{[f(x+h)-f(x-h)]\cos\lambda_k x\}=$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T+h}^{T+h}f(t)\cos\lambda_k(t-h)dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T-h}^{T-h}f(t)\cos\lambda_k(t+h)dt=$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+h)[\cos\lambda_n t\cos\lambda_n h+\sin\lambda_n t\sin\lambda_n h]dt-$$

$$-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t-h)[\cos\lambda_k t\cos\lambda_k h-\sin\lambda_k t\sin\lambda_k h]dt=$$

$$=2\sin\lambda_k h\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)\sin\lambda_k tdt=$$

$$=2\sin\lambda_k hM\{f(t)\sin\lambda_k t\}dt=2b_k\sin\lambda_k h.$$

Similarly, we have repeating these calculations for the coefficients $b_k(F_h)$, we find

$$b_k(F_h) = -2a_k \sin \lambda_k h.$$

Then by virtue of Bessel 's inequality we get

$$\sum_{k=1}^{\infty}(|a_k(f)|^2 + |b_k(f)|^2)\sin^2 \lambda_k h = \sum_{k=1}^{\infty}(|a_k(f)\sin \lambda_k h|^2 + |b_k(f)\sin \lambda_n h|^2) =$$

$$= \frac{1}{4}\sum_{k=1}^{\infty}(|2a_k(f)\sin \lambda_k h|^2 + |2b_k(f)\sin \lambda_k h|^2) = \frac{1}{4}\sum_{k=1}^{\infty}(|a_n(F_h)|^2 + |b_k(F_h)|^2) \leq$$

$$\leq \frac{1}{4}M\{|f(x+h) - f(x-h)|^2\}.$$

For $k \in A_\nu$, consider

$$2^{\nu-1}\pi h \leq \lambda_k h < 2^\nu \pi h.$$

Let's put $h = 2^{-\nu-1}$ and from the latter we get

$$2^{\nu-1}\pi 2^{-\nu-1} \leq \lambda_k h < 2^\nu \pi 2^{-\nu-1}$$

or

$$\frac{\pi}{4} \leq \lambda_k h < \frac{\pi}{2}.$$

Hence,

$$\sin^2 \lambda_k 2^{-\nu-1} \geq \frac{1}{2}.$$

Hence, after using a number of calculations, the inequality (6) follows

$$\sum_{k\in A_\nu}(|a_k(f)|^2 + |b_k(f)|^2) \leq 2\sum_{k\in A_\nu}(|a_k(f)|^2 + |b_k(f)|^2)\sin^2 \lambda_k 2^{-\nu-1} \leq$$

$$\leq 2\sum_{k=1}^{\infty}(|a_k(f)|^2 + |b_k(f)|^2)\sin^2 \lambda_k 2^{-\nu-1} \leq \frac{1}{2}M|f(x+2^{-\nu-1}) - f(x-2^{-\nu-1})|^2.$$

Next, we denote by $\varphi(u)$ the function $\frac{u^2}{\Phi(u)}$, which is non-decreasing. Since the function $f(x)$ is bounded, then $\omega(f, 2^{-\nu}) < \infty$. So, if $\Phi(u) \equiv u^2$, then $\varphi(u) \equiv 1$ and the assumption of the limitation of the function $f(x)$ will be superfluous.

Multiplying and dividing the right part (6) by the function $\Phi[\omega(f, 2^{-\nu})]$, we have

$$\sum_{k\in A_\nu}(|a_k(f)|^2 + |b_k(f)|^2) \leq \frac{1}{2}M\{|f(x+2^{-\nu-1}) - f(x-2^{-\nu-1})|^2\}\frac{\Phi[\omega(f,2^{-\nu})]}{\Phi[\omega(f,2^{-\nu})]} \leq$$

$$\leq 2^{-1}\omega^2(f,2^{-\nu})\Phi^{-1}[\omega(f,2^{-\nu})]\overline{M}\{\Phi[\omega(f,2^{-\nu})]\} = 2^{-1}\omega^2(f,2^{-\nu})\omega_\Phi(f,2^{-\nu})\Phi^{-1}[\omega(2^{-\nu})].$$

Let $m(A_v)$ be a measure of sets in A_v. Then using the Helder inequality from the last inequality we get

$$\sum_{k\in A_\nu}(|a_k(f)|^2+|b_k(f)|^2)^{\frac{\beta}{2}} \leq [m(A_v)]^{1-\frac{\beta}{2}}[\sum_{k\in A_\nu}(|a_k(f)|^2+|b_k(f)|^2)]^{\frac{\beta}{2}} \leq$$

$$\leq [m(A_v)]^{1-\frac{\beta}{2}}[2^{-1}\omega^2(f,2^{-\nu})\omega_\Phi(f,2^{-\nu})\Phi^{-1}[\omega(f,2^{-\nu})]]^{\frac{\beta}{2}} =$$

$$= 2^{-\frac{\beta}{2}}[m(A_v)]^{1-\frac{\beta}{2}}\omega^\beta(f,2^{-\nu})\omega_\Phi^{\frac{\beta}{2}}(f,2^{-\nu})\Phi^{-\frac{\beta}{2}}[\omega(f,2^{-\nu})] \leq$$

$$\leq 2^{-\frac{\beta}{2}}[\mu(2^\nu\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}\omega^\beta(f,2^{-\nu})\omega_\Phi^{\frac{\beta}{2}}(f,2^{-\nu})\Phi^{-\frac{\beta}{2}}[\omega(f,2^{-\nu})]. \qquad (7)$$

So for

$$k_0 = \min_{\lambda_k\geq\pi} k$$

from inequality (7) we find that

$$\sum_{k=k_0}^{\infty}(|a_k(f)|^2+|b_k(f)|^2)^{\frac{\beta}{2}} \leq$$

$$\leq 2^{-\frac{\beta}{2}}\sum_{\nu=1}^{\infty}[\mu(2^\nu\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}\omega^\beta(f,2^{-\nu})\omega_\Phi^{\frac{\beta}{2}}(f,2^{-\nu})\Phi^{-\frac{\beta}{2}}[\omega(f,2^{-\nu})]. \qquad (8)$$

By virtue of condition (4), the series (5) converges. Theorem 1 is proved. □

Theorem 2. *Let the function $f(x)\in B_2$. If at $0<\beta<2$ the condition is met*

$$\sum_{\nu=1}^{\infty}[\mu(2^\nu\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}\omega_2^\beta(f,2^{-\nu}) < \infty,$$

where

$$\omega_2(f,h) = [\sup_{|\delta|\leq h} M\{|f(x+\delta)-f(x)|^2\}]^{\frac{1}{2}},$$

then row (5) converges.

Proof We write inequality (6) in the following form

$$\sum_{k\in A_\nu}(|a_k(f)|^2+|b_k(f)|^2) \leq \frac{1}{2}M\{|f(x+2^{-\nu-1})-f(x-2^{-\nu-1})|^2\} \leq$$

$$\leq \frac{1}{2}\sup_{|\delta|\leq 2^{-\nu}} M\{|f(x+\delta)-f(x)|^2\} = \frac{1}{2}\omega(f,2^{-\nu}).$$

Hence, using the inequality (7), we will have

$$\sum_{k\in A_\nu}(|a_k(f)|^2+|b_k(f)|^2)^{\frac{\beta}{2}} \leq 2^{-\frac{\beta}{2}}[\mu(2^\nu\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}[\omega(f,2^{-\nu})]^{\frac{\beta}{2}} =$$

$$= 2^{-\frac{\beta}{2}}[\mu(2^\nu\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}\omega_2^\beta(f,2^{-\nu}).$$

Then, by the conditions of the theorem, it follows from the latter that it follows that

$$\sum_{k=k_0}^{\infty}(|a_k(f)|^2+|b_k(f)|^2)^{\frac{\beta}{2}} \leq 2^{-\frac{\beta}{2}}\sum_{\nu=1}^{\infty}[\mu(2^{\nu}\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}\omega_2^{\beta}(f,2^{-\nu}) < \infty.$$

Theorem 2 is proved. □

In the future, we will need the following auxiliary statement.

Lemma 1. *If for a non-decreasing function* $\Phi(u) \geq 0$ *at* $u \geq 0$

$$V_{\phi,T}(f) = \sup_{\Pi}\sum_{k=1}^{n}\phi[|f(x_k)-f(x_{k-1})|]_{B_2},$$

where Π *is an arbitrary division of the interval* $(-T;T)$ *by the points* $x_0, x_1, ..., x_N$ *and*

$$V_{\phi}(f) = \overline{M}\{V_{\phi,T}(f)\} = \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}V_{\phi,T}(f)dx,$$

then for any $h>0$ *the following estimate is valid*

$$\overline{M}\{\Phi[f(x+h)-f(x-h)]\} \leq 2hV_{\Phi}(f). \tag{9}$$

Proof Let $V_{\Phi}(f) < \infty$. Suppose that for $\varepsilon > 0$ there exists such a number T_0 that for every $T > T_0$ the inequality holds

$$V_{\Phi,T+3h}(f) \leq 2[V_{\Phi}(f)+\varepsilon](T+3h). \tag{10}$$

Indeed, by definition of the upper limit

$$\frac{1}{2(T+3h)}V_{\Phi,T+3h}(f) \leq V_{\Phi}(f)+\varepsilon,$$

hence the inequality (10).

For a fixed $T > T_0$, we define such an interval $(-T-h, T-h)$ in which the points $x_0, x_1, ..., x_n$ will be

$$x_k - x_{k-1} = 2h \quad (k=1,2,...,n),$$

$$x_k - x_{k-1} \geq 2h \quad (k=n).$$

Then, given the values of $x_k - x_{k-1}$, for up to the limiting average value of the function $\phi[|f(x+h)-f(x-h)|]$ we get

$$\frac{1}{2T}\int_{-T}^{T}\Phi[|f(x+h)-f(x-h)|]dx=\frac{1}{2T}\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\Phi[|f(x+h)-f(x-h)|]dx=$$

$$=\frac{1}{2T}\int_{0}^{2h}\sum_{k=1}^{n}\Phi[|f(x_k+t)-f(x_{k-1}+t)|]dt+$$

$$+\frac{1}{2T}\int_{0}^{x_k-x_{k-1}}\Phi[|f(x_k+t+2h)-f(x_{k-1}+t)|]dt\leq$$

$$\leq\frac{1}{2T}\int_{0}^{2h}\sup_{x_k\in[-T+3h,T+3h]}\sum_{k=1}^{n}\Phi[|f(x_k)-f(x_{k-1})|]dt=$$

$$=\frac{1}{2T}\int_{0}^{2h}V_{\phi,T+h+3h}(f)dt=\frac{1}{2T}V_{\Phi,T+3h}\int_{0}^{2h}dt=$$

$$=\frac{1}{2T}\int_{0}^{2h}V_{\phi,T+h+3h}(f)dt=\frac{h}{T}V_{\Phi,T+3h}\leq$$

$$\frac{2h}{T}[V_{\phi}(f)+\varepsilon][T+3h]=(2h+\frac{6h^2}{T})(V_{\Phi}(f)+\varepsilon),$$

Hence, at $T\to\infty$, going to the limit, we get an estimate (9), which implies the validity of Lemma 1. $\square$

Theorem 3. *Let $f(x)\in B_2$ and a non-decreasing function $\Phi(u)$ is given such that $\Phi(u)>0$ and for $u>0$, $\Phi(0)\geq 0$. If $V_{\Phi}(f)<\infty$ at $0<\beta<2$ and the condition is met*

$$\sum_{\nu=1}^{\infty}[\mu(2^{\nu}\pi)-\mu(2^{\nu-1}\pi)+1]^{1-\frac{\beta}{2}}2^{-\frac{\beta\nu}{2}}\omega^{\beta}(f,2^{-\nu})\Phi^{-\frac{\beta}{2}}[\omega(f,2^{-\nu})]<\infty, \tag{11}$$

then the series (5) converges.

Proof The theorem is proved using Theorem 1 and Lemma 1. Indeed, since

$$\omega_{\Phi}(f,2^{-\nu})=\sup_{|\delta|\leq 2^{-\nu}}\overline{M}\{\Phi[|f(x+\delta)-f(x)|]\}\leq$$

$$\leq 2hV_{\phi}(f)=\sup_{|\delta|\leq 2^{-\nu}}\sum_{\nu=1}^{\infty}\phi[|f(x+2^{-\nu})-f(x)|]\leq 2^{-\nu},$$

then substituting $2^{-\nu}$ instead of $\omega_{\Phi}^{\frac{\beta}{2}}(f,2^{-\nu})$ into inequality (4), we get inequality (8), which proves theorem 3. $\square$

In the future, we will establish the convergence condition of series (5) for the existence of the spectrum $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$. In this case, we need to prove the following

Lemma 2. *If $a_1, a_2, \ldots$ are positive numbers, then for any χ series*

$$\sum_{\nu=1}^{\infty} 2^{\chi\nu} a_{2^\nu}$$

and

$$\sum_{k=1}^{\infty} k^{\chi-1} a_k,$$

either converge or diverge at the same time.

Note that this lemma is implicitly contained in [10] (see page 13).

Indeed, due to the monotony of a_k, for any ν we have

$$2^{\chi\nu} a_{2^\nu} \leq 2^{|\chi|+1} \sum_{k=2^{\nu-1}+1}^{2^\nu} k^{\chi-1} a_k \leq 2^{2|\chi|+1} 2^{\chi(\nu-1)} a_{2^{\nu-1}},$$

and from here

$$\sum_{k=1}^{\infty} k^{\chi-1} a_k = a_1 + \sum_{\nu=1}^{\infty} \sum_{s=2^{\nu-1}+1}^{2^\nu} s^{\chi-1} a_s.$$

Lemma 3. *If $a_k \downarrow 0$ and $\sum_{k=1}^{\infty} a_k = +\infty$, then the flat $\Delta a_k = a_k - a_{k+1}$, we have*

$$\sum_{k=1}^{\infty} k \Delta a_k = +\infty.$$

Let 's put

$$k^{\chi-1} a_k = \sum_{\nu=1}^{k} \nu \Delta a_\nu.$$

By virtue of $a_k \downarrow 0$ we have $\Delta a_\nu \geq 0$ $(\nu = 1, 2, \ldots)$. So all $k^{\chi-^1} a_k \geq 0$ and does not decrease monotonically. We need to prove that $k^{\chi-1} a_k \to \infty$. If this were not true, then

$$k^{\chi-1} a_k \uparrow a \quad (a \neq +\infty).$$

Then $k^{\chi-1} a_k = a - \varepsilon_k$, $\varepsilon_k \downarrow 0$, and hence since

$$k^{\chi-1} a_k - (k-1)^{\chi-1} a_{k-1} = k \Delta a_k = (a - \varepsilon_k) - (a - \varepsilon_{k-1}) = \Delta \varepsilon_{k-1},$$

that

$$\Delta a_k = \frac{\Delta\varepsilon_{k-1}}{k}.$$

By virtue of $a_k \to 0$ and $\Delta\varepsilon_\nu \geq 0$ we have

$$a_k = \sum_{\nu=k}^{\infty} \Delta a_\nu = \sum_{\nu=k}^{\infty} \frac{\Delta\varepsilon_{\nu-1}}{\nu} \leq$$

$$\leq \frac{1}{k}\sum_{\nu=k}^{\infty} \Delta\varepsilon_{\nu-1} = \frac{\varepsilon_{k-1}}{k},$$

and therefore $ka_k \to 0$

But, applying to the sum representing $k^{\chi-1}a_k$, the Abel transform, we find

$$k^{\chi-1}a_k = \sum_{\nu=1}^{k} \nu\Delta a_\nu = \sum_{\nu=1}^{k+1} a_\nu = a_1 + a_2 + ... + a_{k+1},$$

and since $k^{\chi-1}a_k \to a$, and $ka_k \to 0$, then $a_1 + a_2 + ... + a_{k+1} \to 0$, which contradicts the condition

$$\sum_{k=1}^{\infty} a_k = +\infty.$$

Let $\sum_{k=1}^{\infty} k^{\chi-1}a_k$ be a convergent series with $k^{\chi-1}a_k \downarrow 0$. We believe

$$r_n = \sum_{k=1}^{\infty} k^{\chi-1}a_k.$$

We will say that a series satisfies condition (A) if

$$r_n = O(k^{\chi-1}a_k). \tag{12}$$

If the terms of the series decrease no slower than some geometric progression, that is, if

$$(k+1)^{\chi-1}a_{k+1} < \theta k^{\chi-1}a_k, \quad 0 < \theta < 1,$$

then it satisfies condition (A), but the reverse conclusion is, of course, incorrect, as at least such an example shows

$$(2k-1)^{\chi-1}a_{2k-1} - (2k)^{\chi-1}a_{2k} = \theta^k, \quad k \in N; \quad 0 < \theta < 1.$$

We show that if a series satisfies condition (A), then whatever $\theta < 1$, it can be divided into a finite l series (l envy of θ) so that the terms of each of them decrease no slower than the geometric progression with the denominator θ.

Indeed, condition (12) means that $r_n < ck^{\chi-1}a_k$, where c is constant. Let θ be given. Let's choose the number l so that

$$l = [\frac{c}{\theta}]. \tag{13}$$

Then by virtue of the monotonous decreasing of the numbers $k^{\chi-1}a_k$ and by virtue of (13) we have

$$(l+1)(k+l)^{\chi-1}a_{k+l} \leq \sum_{\nu=k}^{\nu=k+l} \nu^{\chi-1}a_\nu \leq \sum_{\nu=k}^{\infty} \nu^{\chi-1}a_\nu \leq ck^{\chi-1}a_k, \tag{14}$$

and therefore it follows from (14) that

$$(k+l)^{\chi-1}a_{k+l} \leq \frac{c}{l+1}k^{\chi-1}a_k \leq \theta k^{\chi-1}a_k.$$

Hence, all l series

$$1^{\chi-1}a_1 + (1+l)^{\chi-1}a_{1+l} + (1+2l)^{\chi-1}a_{1+2l} + \ldots,$$

$$2^{\chi-1}a_2 + (2+l)^{\chi-1}a_{2+l} + (2+2l)^{\chi-1}a_{2+2l} + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$l^{\chi-1}a_l + (2l)^{\chi-1}a_{2l} + (3l)^{\chi-1}a_{3l} + \ldots,$$

it can be decomposed into a series

$$\sum_{\nu=1}^{\infty} \nu^{\chi-1}a_\nu,$$

which decreases no slower than the geometric progression with the denominator θ.

1.2. Let the spectrum be λ_k and for $\rho > 0$ the condition $k^\rho = O(\lambda_k)$ is satisfied. Let's choose an increasing function $\lambda(x)$ such that $\lambda(k) = \lambda_k$ then

$$y^\rho = x\frac{y^\rho}{\lambda(y)} = O(x) \quad (x = \lambda(y)),$$

hence

$$y = O\{x^{\frac{1}{\rho}}\}.$$

Hence

$$\mu(2^\nu\pi) - \mu(2^{\nu-1}\pi) \le \mu(2^\nu\pi) + 1 = O\{2^{\frac{\nu}{\rho}}\},$$

and condition (4) can be replaced by the condition

$$\sum_{\nu=1}^{\infty} 2^{\frac{(1-\beta)\nu}{\rho}} \omega^\beta(f, 2^{-\nu}) \omega_\Phi^{\beta/2}(f, 2^{-\nu}) \Phi^{-\beta/2}[\omega(f, 2^{-\nu})] < \infty,$$

или согласно леммы 2

$$\sum_{\nu=1}^{\infty} k^{\frac{(1-\beta)\nu}{\rho}-1} \omega^\beta(f, k^{-1}) \omega_\Phi^{\beta/2}(f, k^{-1}) \Phi^{-\beta/2}[\omega(f, k^{-1})] < \infty.$$

The statement of Lemma 2 is obtained from the fact that the function $\frac{u^2}{\Phi(u)}$ is non-decreasing. By virtue of Theorem 2 , the following holds.

Theorem 4. *Let* $k^\rho = O(\lambda_k)$ *when* $\rho > 0$. *And let for the function* $f \in B_2$ *at* $0 < \beta < 2$ *takes place*

$$\sum_{k=1}^{\infty} k^{\frac{1-\frac{\beta}{2}}{\rho}-1} \omega_2^\beta(f, k^{-1}) < \infty, \tag{15}$$

where $\omega_2(f, h) = [\sup_{|\delta|\le h} M\{|f(x+\delta) - f(x)|^2\}]^{\frac{1}{2}}$. *Then the series (7) converges.*

For $\rho = 1$ and $\omega(h) = O\{h^\alpha\}$, the result is obtained in $\beta > \frac{2}{2+\alpha(2-r)}$ (see [8], p. 137), and for $\rho = \beta = 1$ we get Bernstein (see [7], p. 231)

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} \omega_2(f, k^{-1}) < \infty.$$

Now we denote $V_r(f) = [V_\Phi(f)]^{1/r}$ for $\Phi(u) = u^r$. The value of $V_r(f)$ is called r-a variation of the function $f(x)$.

According to Theorem 3 and Lemma 2 , the following statement holds

Theorem 5. *Let* $k^\rho = O(\lambda_k)$ *when* $\rho > 0$. *If at* $0 < r \le 2$ *the function* $f \in B_2$ *has a finite* r*-variation, besides at* $0 < \beta < 2$ *exists*

$$\sum_{k=1}^{\infty} k^{\frac{1-\frac{\beta}{2}}{\rho} - \frac{\beta}{2} - 1} \omega_n^{\beta(1-\frac{r}{2})}(f, k^{-1}) < \infty,$$

then the series (5) converges.

In the case when $\rho = 1$ and $\omega(h) = O\{h^\alpha\}$ we get $\beta > \frac{2}{2+\alpha(2-r)}$, which for $r = 1$ obtained by Varashkevich and Zygmunde (see [8], p. 138). For $\rho = \beta = 1$, we obtain the following statement about the absolute convergence of the series (3).

Theorem 6. *Let $f \in B_2$ and $k^\rho = O(\lambda_k)$ when $\rho > 0$. Suppose that $\omega(h) = O\{h^\alpha\}$ and $V_r(f) < \infty$ for $\alpha > 0$, $0 < r \leq 2$. Then the series (3) absolutely converges.*

This theorem for $r = 1$ was obtained by Sigmund. If in Theorem 5 $\rho = \beta = r = 1$ we obtain the known condition of absolute convergence of series (3) ([7], p. 231)

$$\sum_{k=1}^{\infty} \frac{1}{k}\sqrt{\omega(f, k^{-1})} < \infty,$$

which generalizes Sigmund's result.

1.3. **Definition 3.** A sequence of natural numbers

$$n_1 < n_2 < ... < n_k < ...$$

are called lacunar if there exists such a $q > 1$ that

$$\frac{n_{k+1}}{n_k} \geq \quad (k = 1, 2, ...).$$

Now let the lacunar condition

$$\frac{\lambda_{k+1}}{\lambda_k} > q > 1$$

be satisfied for the spectrum $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$. Then the spectrum $\frac{\lambda_k}{q^k}$ will be increasing. Let's choose such a function $\lambda(x)$ so that the function $\frac{\lambda(x)}{q^x}$. was increasing and $\lambda(k) = \lambda_k$. Then also increasing and denoting $y = \mu(x)$ we have

$$y = log_q x - log_q \frac{\lambda(y)}{q^{y-1}} + 1,$$

then in the case of $y_1 = \mu(2^{\nu-1}\pi)$, $y_2 = \mu(2^\nu \pi)$ we have

$$\mu(2^\nu\pi) - \mu(2^{\nu-1}\pi) + 1 = log_q 2 - log_q \frac{\lambda(y_2) q^{y_1}}{q^{y_2}\lambda(y_1)} + 1 \leq$$
$$\leq log_q 2 + 1 = O\{1\} \qquad (16)$$

By virtue of Lemma 2, condition (6) can be replaced by the condition

$$\sum_{k=1}^{\infty} \frac{1}{k} \omega^\beta(f, \frac{1}{k}) \omega_\Phi^{\frac{\beta}{2}}(f, \frac{1}{k}) \Phi^{-\frac{\beta}{2}}[\omega(f, \frac{1}{k})] < \infty.$$

Theorem 7. *Let the function $f \in B_2$ be bounded at $\frac{\lambda_{k+1}}{\lambda_k} > q > 1$. And let the condition of Theorem 1 be satisfied for the function $\phi(u)$, for $\alpha > 0$, $\omega_\phi(h = O\{h^\alpha\})$. Then for each $0 < \alpha < 2$ the series (5) converges.*

Theorem 8. *Let $\frac{\lambda_{k+1}}{\lambda_k} > q > 1$. Then if $f \in B_2$ and for $\alpha > 0$ the condition $\omega_2(h = O\{h^\alpha\})$ is satisfied, then for $0 < \alpha < 2$ the series (5) converges.*

By virtue of Theorem 3 we get:

Theorem 9. *Let $\frac{\lambda_{k+1}}{\lambda_k} > q > 1$ and the function $\phi(u)$ satisfy the conditions of Theorem 3. Let the function $f \in B_2$ be bounded and $V_\phi(f) < \infty$. Then at $0 < \alpha < 2$ the series (5) converges.*

For proofs, it is sufficient to note that the condition

$$\sum_{\nu=1}^{\infty} 2^{\frac{-\beta\nu}{2}} \omega^\beta(f, 2^{-\nu}) \Phi^{-\frac{\beta}{2}}[\omega(f, 2^{-\nu})] < \infty \tag{17}$$

follows when substituting (16) for (11). However, according to Lemma 2, the condition (17) is equivalent to the condition

$$\sum_{\nu=1}^{\infty} \frac{1}{k^{1+\frac{\beta}{2}}} \omega^\beta(f, \frac{1}{k}) \Phi^{-\frac{\beta}{2}}[\omega(f, 2^{-\nu})] < \infty,$$

which follows from the limitations of the spectrum

$$\omega^\beta(f, \frac{1}{k}) \Phi^{-\frac{\beta}{2}}[\omega(f, 2^{-\nu})]$$

and is performed when $\beta > 0$.

When $f(x)$ is a bounded function, theorems 7, 8, 9 at $\beta \geq 1$ are weaker than Sidon's results stating that the lacunar series of periodic and bounded absolute functions converge (see [8], page 139), for almost-periodic functions [6].

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