Products of Multiplication, Composition and Differentiation on Weighted Bergman Spaces on the Unit Ball

Chao Zhang

ABSTRACT

We define differentiation operator on H(B) by radial derivative, then we study the boundedness and compactness of products of multiplication, composition and differentiation on weighted Bergman spaces on the unit ball.

Keywords and Phrases: composition operator; multiplication operator; differentiation operator; weighted bergman space.

Classification: LCC Code: QA331.7

Language: English

Great Britain Journals Press

LJP Copyright ID: 925603
Print ISSN: 2631-8490
Online ISSN: 2631-8504

London Journal of Research in Science: Natural and Formal

Volume 24 | Issue 1 | Compilation 1.0

Products of Multiplication, Composition and Differentiation on Weighted Bergman Spaces on the Unit Ball

Chao Zhang

ABSTRACT

We define differentiation operator on H(B) by radial derivative, then we study the boundedness and compactness of products of multiplication, composition and differentiation on weighted Bergman spaces on the unit ball.

Keywords and Phrases: composition operator; multiplication operator; differentiation operator; weighted bergman space.

I. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball of $\mathbb{C}^n$, and $\mathbb{S} = \partial\mathbb{B}$ its boundary. We will denote by dv the normalized Lebesgue measure on $\mathbb{B}$.

Recall that for $\alpha > -1$ the weighted Lebesgue measure dv_α is defined by

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z),$$

where

$$c_\alpha = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}$$

is a normalizing constant so that dv_α is a probability measure on $\mathbb{B}$.

Let $\mathbf{H}(\mathbb{B})$ denotes the space of holomorphic functions on $\mathbb{B}$. Take $1 \le p < \infty$.

Then $f \in \mathbf{H}(\mathbb{B})$ is said to be in the weighted Bergman space $\mathbf{A}^p_\alpha(\mathbb{B})$ if

$$\|f\|^p_{\mathbf{A}^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p dv_\alpha(z) < \infty.$$

Let φ be an analytic self-mapping of $\mathbb{B}$, then the composition operator on $\mathbf{H}(\mathbb{B})$ is given by

$$C_\varphi f = f \circ \varphi.$$

Recently, there have been an increasing interest in studying composition operators acting on different spaces of analytic functions, for example, see [2,3] for details about composition operators on classical spaces of analytic functions.

Let D be the differentiation operator defined by

$$Df = f', \quad f \in \mathbf{H}(\mathbb{D}).$$

Hibschweiler and Portnoy [3] defined the linear operators DC_φ and $C_\varphi D$ and investigated the boundedness and compactness of these operators between Bergman

London Journal of Research in Science: Natural and Formal

spaces using Carleson-type measure. S. Ohno [4] discussed boundedness and compactness of $C_\varphi D$ between Hardy spaces. Recall the multiplication operator M_ψ defined by

$$M_\psi f = \psi f, \quad f \in \mathbf{H}(\mathbb{D}).$$

A. K. Sharma defined [5] products of these operators in the following six ways:

$$\begin{aligned}
(M_\psi C_\varphi Df)(z) &= \psi(z) f'(\varphi(z)),\\
(M_\psi DC_\varphi f)(z) &= \psi(z)(\varphi'(z)) f'(\varphi(z)),\\
(C_\varphi M_\psi Df)(z) &= \psi(\varphi(z)) f'(\varphi(z)),\\
(DM_\psi C_\varphi f)(z) &= \psi'(z) f(\varphi(z)) + \psi(z)(\varphi'(z)) f'(\varphi(z)),\\
(C_\varphi DM_\psi f)(z) &= \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),\\
(DC_\varphi M_\psi f)(z) &= \psi'(\varphi(z)) f(\varphi(z))\varphi'(z) + \psi(\varphi(z)) f'(\varphi(z))\varphi'(z).
\end{aligned}$$

for $z \in \mathbb{D}$ and $f \in \mathbf{H}(\mathbb{D})$.

There are a lot of papers researching these products, see [6,7,8]. Since those results focus on $\mathbb{D}$, naturally, we consider similar questions on $\mathbb{B}$. Of course, the method we used is different from the case on $\mathbb{D}$.

For $f \in \mathbf{H}(\mathbb{B})$, we define the differentiation operator on $\mathbf{H}(\mathbb{B})$ by radial derivative. Recall that for $z \in \mathbb{B}$ and $f \in \mathbf{H}(\mathbb{B})$,

$$Rf = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \lim_{r \to 0} \frac{f(z + rz) - f(z)}{r}, \quad r \in \mathbb{R}.$$

One can see that for $z \neq \varphi^{-1}(0)$,

$$|R(f \circ \varphi)(z)| = \frac{|(Rf)(\varphi(z)) \cdot R\varphi(z)|}{|\varphi(z)|}.$$

Then we also have six ways of products of these operators on the unit ball:

$$\begin{aligned}
(M_\psi C_\varphi R)f(z) &= \psi(z) \cdot (Rf)(\varphi(z)),\\
(C_\varphi M_\psi Rf)(z) &= \psi(\varphi(z)) \cdot (Rf)(\varphi(z)),\\
|(M_\psi RC_\varphi f)(z)| &= \frac{|\psi(z) \cdot R\varphi(z) \cdot (Rf)(\varphi(z))|}{|\varphi(z)|},\\
(C_\varphi RM_\psi f)(z) &= (R\psi)(\varphi(z)) \cdot f(\varphi(z)) + \psi(\varphi(z)) \cdot (Rf)(\varphi(z)),\\
(RM_\psi C_\varphi f)(z) &= f(\varphi(z)) \cdot R\psi(z) + R(f(\varphi(z))),\\
(RC_\varphi M_\psi f)(z) &= R(\quad(\varphi(z))) \cdot f(\varphi(z)) + R(f(\varphi(z))) \cdot \psi(\varphi(z))
\end{aligned}$$

for $z \neq \varphi^{-1}(0)$.

In this paper, we characterize the boundedness and compactness of $M_\psi RC_\varphi$, $M_\psi C_\varphi R$ and $RC_\varphi M_\psi$ on the weighted Bergman spaces on the unit ball.

2. $M_\psi RC_\varphi$

For $a, b \in \mathbb{B}$, we will denote $\beta(a, b)$ the distance with the Bergman metric on $\mathbb{B}$. For $r > 0$, let the Bergman metric ball

$$D(a, r) = \{z \in \mathbb{B} : \ \beta(a, z) < r\}.$$

For a point $\zeta \in \mathbb{S}$ and $t > 0$, the non-isotropic metric ball with center ζ and radius t is

$$Q_t(\zeta) = \{z \in \mathbb{B} : |1 - \langle z, \zeta \rangle| < t\}.$$

The following Lemma is Theorm 50 of [9].

Lemma 2.1 Suppose $0 < p \leq q < \infty$, α is real, and λ is a positive Borel maesure on $\mathbb{B}$. Then for any nonnegative integer m with $\alpha + mp > -1$ the following conditions are equivalent.

(a) There is a constant $C > 0$ such that

$$\int_{\mathbb{B}} |R^m f(w)|^q d\lambda(w) \leq C \|f\|^q_{\mathbf{A}^p_\alpha}$$

for all $f \in \mathbf{A}^p_\alpha(\mathbb{B})$.

(b) For each (or some) $s > 0$ there is a constant $C > 0$ such that

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^s}{|1 - \langle z, w \rangle|^{s + (n+1+\alpha+mp)q/p}} d\lambda(w) \leq C$$

for all $z \in \mathbb{B}$.

(c) There is a constant $C > 0$ such that

$$\lambda(Q_t(\zeta)) \leq C t^{(n+1+\alpha+mp)q/p}$$

for all $t > 0$ and $\zeta \in \mathbb{S}$.

(d) For each (or some) $r > 0$ there is a constant $C > 0$ such that

$$\lambda(D(a, r)) \leq C(1 - |a|^2)^{(n+1+\alpha+mp)q/p}$$

for all $a \in \mathbb{B}$.

Theorem 2.2. *Let $0 < p \leq q$ and $\alpha, \beta > -1$. Let φ, ψ be a holomorphic maps on $\mathbb{B}$ and $\frac{\psi R\varphi}{|\varphi|} \in \mathbf{A}^q_\beta(\mathbb{B})$. Define a finite positive Borel measure μ on $\mathbb{B}$ by*

$$\mu(E) = \int_{\varphi^{-1}(E)} \left(\frac{|\psi(z) \cdot R\varphi(z)|}{|\varphi(z)|} \right)^q dv_\beta(z)$$

for all Borel sets E of $\mathbb{B}$. Then the following are equivalent:

(1) $M_\psi RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$.

(2)

$$\mu(D(a, r)) = O((1 - |a|^2)^{\frac{q(n+1+\alpha+p)}{p}} \text{ as } |a| \to 1.$$

Proof. Suppose (1) holds. Since $\frac{\psi R\varphi}{|\varphi|} \in \mathbf{A}^q_\beta(\mathbb{B})$, by the definition of μ, we get (see [10, p.163])

$$\begin{aligned}\|M_\psi RC_\varphi(f)\|^q_{\mathbf{A}^q_\beta} &= \int_{\mathbb{B}} \Big(\frac{|\psi(z)\cdot(Rf)(\varphi(z))\cdot R\varphi(z)|}{|\varphi(z)|}\Big)^q dv_\beta(z)\\ &= \int_{\mathbb{B}} |Rf(w)|^q d\mu_(w)\\ &= \|Rf\|^q_{\mathbf{L}^q(\mu)}.\end{aligned}$$

Since $M_\psi RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$,

$$\|Rf\|^q_{\mathbf{L}^q(\mu)} = \|M_\psi RC_\varphi(f)\|^q_{\mathbf{A}^q_\beta} \le C\|f\|^q_{\mathbf{A}^p_\alpha}$$

holds for all $f \in \mathbf{A}^p_\alpha(\mathbb{B})$. From Lemma 2.1, one can see that

$$\mu(D(a,r)) = O((1-|a|^2)^{\frac{q(n+1+\alpha+p)}{p}} as\ |a| \to 1.$$

Conversely, if (2) holds, also by Lemma 2.1, we have

$$\|M_\psi RC_\varphi(f)\|^q_{\mathbf{A}^q_\beta} = \|Rf\|^q_{\mathbf{L}^q(\mu)} \le C\|f\|^q_{\mathbf{A}^p_\alpha}.$$

Then, $M_\psi RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$.

The following lemmas were obtained in [11] and [9] respectively.

Lemma 2.3. let $r > 0$, $p > 0$, $\alpha > -1$, then there is a constant C such that

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in \mathbf{H}(\mathbb{B})$ and all $z \in \mathbb{B}$.

Lemma 2.4. Suppose $p > 0$, $n+1+\alpha > 0$, then there exists a constant $C > 0$ (depending on p and α) such that

$$|f(z)| \le \frac{C\|f\|_{\mathbf{A}^p_\alpha}}{(1-|z|^2)^{\frac{n+1+\alpha}{p}}}$$

for all f in $\mathbf{A}^p_\alpha(\mathbb{B})$ and $z \in \mathbb{B}$.

Theorem 2.5. Let $0 < p \le q$ *and* $\alpha, \beta > -1$. *Let* φ, ψ be a holomorphic maps on $\mathbb{B}$ and $\frac{\psi R\varphi}{|\varphi|} \in \mathbf{A}^q_\beta(\mathbb{B})$. Define a finite positive Borel measure μ on $\mathbb{B}$ by

$$\mu(E) = \int_{\varphi^{-1}(E)} \Big(\frac{|\psi(z)\cdot R\varphi(z)|}{|\varphi(z)|}\Big)^q dv_\beta(z)$$

for all Borel sets E of $\mathbb{B}$. Then the following are equivalent:

(1) $M_\psi RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$.

(2)
$$\mu(D(a,r)) = o((1-|a|^2)^{\frac{q(n+1+\alpha+p)}{p}} as\ |a| \to 1.$$

Proof. First suppose that $M\psi\, RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$. Let $a \in \mathbb{B}$ and consider function

$$f_a(z) = \frac{(1-|a|^2)^{\frac{n+1+\alpha}{p}}}{(1-\langle z,a\rangle)^{\frac{2(n+1+\alpha)}{p}}}.$$

Clearly $\|f_a\|_{\mathbf{A}^p_\alpha} \cong 1$ and f_a converges to zero uniformly on compact subsets of $\mathbb{B}$ as $|a| \to 1$. Since $M\psi\, RC_\varphi$ is compact, so for gives $\varepsilon > 0$, we can find $0 < r_0 < 1$ such that $\|M\psi RC_\varphi(f)\|^q_{\mathbf{A}^p_\alpha} < \varepsilon$ for $|a| > r_0$. Thus

$$\varepsilon > \int_{\mathbb{B}} |Rf_a(z)|^q d\mu(z) \geq \int_{D(a,r)} |Rf_a(z)|^q d\mu(z)$$

for $|a| > r_0$. Since $1-|a|^2 \cong |1-\bar{a}z|$ when $z \in D(a,r)$, so

$$|Rf_a(z)| = \frac{2(n+1+\alpha)(1-|a|^2)^{\frac{n+1+\alpha}{p}}\langle z,a\rangle}{p(1-\bar{a}z)^{\frac{2(n+1+\alpha)+p}{p}}} \cong \frac{2(n+1+\alpha)|a|^2}{p(1-|a|^2)^{\frac{n+1+\alpha+p}{p}}}.$$

Then

$$\mu(D(a,r)) = o((1-|a|^2)^{\frac{q(n+1+\alpha+p)}{p}}$$

as $|a| \to 1$.

Conversely, assume that (2) holds. Let $\{f_k\}$ be a sequence in $\mathbf{A}^p_\alpha(\mathbb{B})$ such that $\|f_k\|_{\mathbf{A}^p_\alpha} \leq M\psi$ and $\{f_k\} \to 0$ uniformly on compact subsets of $\mathbb{B}$. To show that $M\ \ RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$, it is sufficient to prove that

$$\|M\psi\, RC_\varphi(f_k)\|^q_{\mathbf{A}^q_\beta} = \|Rf_k\|^q_{L^q(\mu)} \to 0 \ as\ k \to \infty$$

From Lemma 2.3,

$$\int_{\mathbb{B}} |Rf_k|^q d\mu \leq C \int_{\mathbb{B}} \frac{1}{(1-|a|^2)^{n+1+\alpha}} \int_{D(a,r)} |Rf_k(z)|^q dv_\alpha(z) d\mu(a).$$

Note that $\chi_{D(a,r)}(z) = \chi_{D(z,r)}(a)$ and $1-|a|^2 \cong 1-|z|^2$ when $a \in D(z,r)$. At the same time, $f_k \in \mathbf{A}^p_\alpha(\mathbb{B})$ if and only if $Rf_k \in \mathbf{A}^p_{\alpha+p}(\mathbb{B})$, then by lemma 2.4,

$$|Rf_k(z)| \leq \frac{\|Rf_k\|_{\mathbf{A}^p_{\alpha+p}}}{(1-|z|^2)^{\frac{n+1+\alpha+p}{p}}} \leq \frac{C\|f_k\|_{\mathbf{A}^p_\alpha}}{(1-|z|^2)^{\frac{n+1+\alpha+p}{p}}}.$$

Then, by an application of Fubini's theorem, we have

$$\begin{aligned}
\|M\psi\, RC_\varphi(f)\|^q_{\mathbf{A}^q_\beta} &\leq C' \int_{\mathbb{B}} |Rf_k(z)|^q \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} dv_\alpha(z) \\
&\leq C' \|f_k\|^{q-p}_{\mathbf{A}^p_\alpha} \int_{\mathbb{B}} |Rf_k(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{\frac{q(n+1+\alpha+p)-p^2}{p}}} dv_\alpha(z) \\
&\leq C' M^{q-p}\Big(\int_{|z|\leq r_0} |Rf_k(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{\frac{q(n+1+\alpha+p)-p^2}{p}}} dv_\alpha(z) \\
&+ \int_{|z|>r_0} |Rf_k(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{\frac{q(n+1+\alpha+p)-p^2}{p}}} dv_\alpha(z)\Big) \\
&= I + II.
\end{aligned}$$

Now (2) implies that for a give $\varepsilon > 0$, there is $0 < r_0 < 1$ such that

$$\begin{aligned} II &= C'M^{q-p}\int_{|z|>r_0} |Rf_k(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{\frac{q(n+1+\alpha+p)-p^2}{p}}} dv_\alpha(z) \\ &\leq \varepsilon C'M^{q-p}\int_{|z|>r_0} |Rf_k(z)|^p (1-|z|^2)^p dv_\alpha(z) \\ &\leq \varepsilon C'M^{q-p}\|f_k\|^p_{\mathbf{A}^p_\alpha} \\ &\leq \varepsilon C'M^q. \end{aligned}$$

Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$,

$$\begin{aligned} I &= C'M^{q-p}\int_{|z|\leq r_0} |Rf_k(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{\frac{q(n+1+\alpha+p)-p^2}{p}}} dv_\alpha(z) \\ &\leq \varepsilon C_1 C'M^{q-p}\int_{\mathbb{B}} \mu(D(z,r)) dv_\alpha(z) \\ &\leq \varepsilon C_1 C_2 C'M^{q-p}\int_{\mathbb{B}} \mu(\mathbb{B}) dv_\alpha(z) \\ &= \varepsilon C_1 C_2 C_3 C'M^{q-p}. \end{aligned}$$

for k large enough. Thus

$$\lim_{n\to\infty} \|M\psi\, RC_\varphi f_k\|^q_{\mathbf{A}^q_\beta} = 0,$$

and hece $M\ \ RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$.

Lemma 2.6. [9, Theorem 54] Let $0 < p < q < \infty$ *and* α be any real number, and let λ be a positive Borel measure on $\mathbb{B}$. Then for any nonnegative integer m with $\alpha + mp > -1$ the following conditions are equivalent.

(a) There is a constant $C > 0$ such that

$$\int_{\mathbb{B}} |R^m f(w)|^q d\mu(w) \leq C\|f\|^q_{\mathbf{A}^p_\alpha}$$

for all $f \in \mathbf{A}^p_\alpha(\mathbb{B})$.

(b) For any bounded sequence $\{f_j\}$ in $\mathbf{A}^p_\alpha(\mathbb{B})$ with $f_j(z) \to 0$ for every $z \in \mathbb{B}$,

$$\lim_{j\to\infty} \int_{\mathbb{B}} |R^m f_j(z)|^q d\lambda(z) = 0.$$

(c) For any fixed $r > 0$, define the function

$$\widehat{\lambda}(z) = \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha+mp}}, \quad z \in \mathbb{B},$$

then $\widehat{\lambda}(z) \in \mathbf{L}^{\frac{p}{p-q}}(v_{\alpha+mp})$.

(d) For any fixed $s > 0$, define the function

$$B(\lambda)(z) = \int_{\mathbb{B}} \frac{(1-|z|^2)^s d\lambda(w)}{|1-\langle z,w\rangle|^{n+1+s+mp}}, \quad z \in \mathbb{B},$$

then $B(\lambda)(z) \in \mathbf{L}^{\frac{p}{p-q}}(v_{\alpha+mp})$.

(d) For any fixed $s > 0$, define the function

$$B(\lambda)(z) = \int_{\mathbb{B}} \frac{(1-|z|^2)^s d\lambda(w)}{|1-\langle z,w\rangle|^{n+1+s+mp}}, \quad z \in \mathbb{B},$$

then $B(\lambda)(z) \in \mathbf{L}^{\frac{p}{p-q}}(v_{\alpha+mp})$.

Theorem 2.7. Let $0 < p \le q$ *and* $\alpha, \beta > -1$. *Let* φ, ψ be a holomorphic maps on $\mathbb{B}$ and $\frac{\psi R\varphi}{|\varphi|} \in \mathbf{A}^q_\beta(\mathbb{B})$. Define a finite positive Borel measure μ on $\mathbb{B}$ by

$$\mu(E) = \int_{\varphi^{-1}(E)} \Big(\frac{|\ \ (z)\cdot R\varphi(z)|}{|\varphi(z)|}\Big)^q dv_\beta(z)$$

for all Borel sets E of $\mathbb{B}$. Let $G(z) = \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha+p}}$. Then the following are equivalent:

(1) $M\psi\, RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$.
(2) $M\psi\, RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$.
(3) $G \in \mathbf{L}^{\frac{p}{p-q}}(v_{\alpha+p})$.

Proof. (1)$\Longleftrightarrow$(3). Suppose (1) holds. By the computation before,

$$\|M\psi\, RC_\varphi f\|^q_{\mathbf{A}^q_\beta} = \|Rf\|^q_{\mathbf{L}^q(\mu)}.$$

Since $M\psi\, RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$, we can find a positive constant C such that

$$\|Rf\|^q_{\mathbf{L}^q(\mu)} \le C\|f\|^q_{\mathbf{A}^p_\alpha}.$$

Then by Lemma 2.1 and Lemma 2.6, $M\psi\, RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$ if and only if $G \in \mathbf{L}^{\frac{p}{p-q}}(v_{\alpha+p})$.

It is clear that (2) implies (1).

It remains to verify that (3) implies (2). Assume that

$$\|f_k\|_{\mathbf{A}^p_\alpha} \le C$$

and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$. It is sufficient to show that

$$\lim_{n\to\infty} \|M\psi\, RC_\varphi f_k\|^q_{\mathbf{A}^q_\beta} = 0.$$

By the computation in the Theorem 2.5, we have

$$\begin{aligned}\|M\psi\, RC_\varphi f_k\|^q_{\mathbf{A}^q_\beta} &\le C\int_{\mathbb{B}} |Rf_k(z)|^q \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} dv_\alpha(z)\\ &= C\int_{\mathbb{B}} |Rf_k(z)|^q G(z) dv_{\alpha+p}(z).\end{aligned}$$

Let $\varepsilon > 0$. Then the hypothesis of (3) implies that there exists $0 < r_0 < 1$ such that

$$\int_{|z|>r_0} (G(z))^{\frac{p}{p-q}} dv_{\alpha+p}(z) < \varepsilon^{\frac{p}{p-q}}.$$

It follows by Holder's inequality that

$$\begin{aligned} & \int_{|z|>r_0} |Rf_k(z)|^q G(z) dv_{\alpha+p}(z) \\ \leq & \left(\int_{\mathbb{B}} |Rf_k(z)|^p dv_{\alpha+p}(z)\right)^{\frac{q}{p}} \left(\int_{|z|>r_0} (G(z))^{\frac{p}{p-q}} dv_{\alpha+p}(z)\right)^{\frac{p-q}{p}} \\ \leq & \varepsilon \|Rf_k\|^q_{\mathbf{A}^p_{\alpha+p}} \\ \leq & \varepsilon C \|f_k\|^q_{\mathbf{A}^p_{\alpha}} \\ \leq & C\varepsilon. \end{aligned}$$

Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$, by Cauchy's estimate, $|Rf_k| < \varepsilon$ for all $|z| < r_0$ and for all $n > n_0$. Thus

$$\int_{|z|\leq r_0} |Rf_k(z)|^q G(z) dv_{\alpha+p}(z) \leq \varepsilon^q \int_{|z|\leq r_0} G(z) dv_{\alpha+p}(z).$$

for all $n > n_0$. Since $\frac{\psi R\varphi}{\varphi} \in \mathbf{A}^q_\beta(\mathbb{B})$ and thus

$$G(z) \leq C\mu(D(z,r)) \leq C\mu(\mathbb{B}) < \infty$$

thus

$$\int_{|z|\leq r_0} G(z) dv_{\alpha+p}(z) \leq C \int_{\mathbb{B}} \mu(D(z,r)) dv_{\alpha+p}(z) \leq C.$$

Then

$$\int_{|z|\leq r_0} |Rf_k(z)|^q G(z) dv_{\alpha+p}(z) \leq C\varepsilon$$

for $n > n_0$. Hence, $M \;\; RC_\varphi$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$.

3. $M_\psi C_\varphi R$

Similar to the proof in section 2, we have the following results about $M_\psi C_\varphi R$, here we omit the details.

Theorem 3.1. Let $0 < p \leq q$ *and* $\alpha, \beta > -1$. *Let* φ, ψ be a holomorphic maps on $\mathbb{B}$ and $\psi\varphi \in \mathbf{A}^q_\beta(\mathbb{B})$. Define a finite positive Borel measure μ on $\mathbb{B}$ by

$$\mu(E) = \int_{\varphi^{-1}(E)} | \;\; (z)|^q dv_\beta(z)$$

for all Borel sets E of $\mathbb{B}$. Then the following are equivalent:

(1) $M_\psi C_\varphi R$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$.

(2)

$$\mu(D(a,r)) = O((1-|a|^2)^{\frac{q(n+1+\alpha+p)}{p}} \; as \; |a| \to 1.$$

Theorem 3.2. Let $0 < p \leq q$ *and* $\alpha, \beta > -1$. *Let* φ, ψ be a holomorphic maps on $\mathbb{B}$ and $\psi\varphi \in \mathbf{A}^q_\beta(\mathbb{B})$. Define a finite positive Borel measure μ on $\mathbb{B}$ by

$$\mu(E) = \int_{\varphi^{-1}(E)} | \quad (z)|^q dv_\beta(z)$$

for all Borel sets E of $\mathbb{B}$. Then the following are equivalent:

(1) $M \quad C_\varphi R$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$.
(2)

$$\mu(D(a,r)) = o((1-|a|^2)^{\frac{q(n+1+\alpha+p)}{p}} as\ |a| \to 1.$$

Theorem 3.3. Let $0 < p \leq q$ *and* $\alpha, \beta > -1$. *Let* φ, ψ be a holomorphic maps on $\mathbb{B}$ and $\psi\varphi \in \mathbf{A}^q_\beta(\mathbb{B})$. Define a finite positive Borel measure μ on $\mathbb{B}$ by

$$\mu(E) = \int_{\varphi^{-1}(E)} | \quad (z)|^q dv_\beta(z)$$

for all Borel sets E of $\mathbb{B}$. Let $G(z) = \frac{\mu^{(D(\ _{z,r}))}}{(1-|z|^2)^{n+1+\alpha+p}}$. Then the following are equivalent:

(1) $M\psi\, C_\varphi R$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ boundedly into $\mathbf{A}^q_\beta(\mathbb{B})$.
(2) $M\psi\, C_\varphi R$ maps $\mathbf{A}^p_\alpha(\mathbb{B})$ compactly into $\mathbf{A}^q_\beta(\mathbb{B})$.
(3) $G \in \mathbf{L}^{\frac{p}{p-q}}(v_{\alpha+p})$.

4. $RC_\varphi M_\psi$

In this section, we characterize the boundedness and compactness of $RC_\varphi M\psi$ by using Carleson measures.

Recall that a positive Borel measure μ on $\mathbb{B}$ is called Carleson measure for $\mathbf{A}^p_\alpha(\mathbb{B})$ if there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}} |f|^p d\mu \leq C \int_{\mathbb{B}} |f|^p dv_\alpha$$

for all $f \in \mathbf{A}^p_\alpha(\mathbb{B})$.

Similarly, a positive Borel measure μ on $\mathbb{B}$ is called a vanishing Carleson measure for $\mathbf{A}^p_\alpha(\mathbb{B})$ if

$$\lim_{k\to\infty} \int_{\mathbb{B}} |f_k|^p d\mu = 0$$

whenever $\{f_k\}$ is a bounded sequence in $\mathbf{A}^p_\alpha(\mathbb{B})$ that converges to 0 uniformly on compact subsets of $\mathbb{B}$.

Theorem 4.1. Let $1 \leq p < \infty$, $\alpha > -1$. Let φ be a holomorphic self-map of $\mathbb{B}$ with $\frac{R\varphi}{|\varphi|} \in \mathbf{A}^p_\alpha(\mathbb{B})$ and $\quad \in \mathbf{A}^p_\alpha(\mathbb{B})$ such that $R\psi \in \mathbf{A}^p_\alpha(\mathbb{B})$. Define a finite positive Borel measure $\mu_{\varphi,\alpha}$ on $\mathbb{B}$ by

$$\mu_{\varphi,\alpha}(E) = \int_{\varphi^{-1}(E)} \Big(\frac{|R\varphi(z)|}{|\varphi(z)|}\Big)^p dv_\alpha(z)$$

for all Borel sets E of $\mathbb{B}$. Let $d\mu(w) = |\psi(\omega)|^p d\mu_{\varphi,\alpha}(w)$. If for every (or some) $r > 0$, there is a constant $C > 0$ such that

$$\mu(D(a,r)) \leq C(1-|a|^2)^{n+1+\alpha+p} \tag{1}$$

holds for all $a \in \mathbb{B}$, then $RC_{\varphi}M_{\psi}$ is bounded on $\mathbf{A}^p_\alpha(\mathbb{B})$ if and only if $|R\psi|^p d\mu_{\varphi,\alpha}$ is a Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$.

Proof. First suppose that $|R\psi|^p d\mu$ is a Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$. Then for $f \in \mathbf{A}^p_\alpha(\mathbb{B})$, by the definition of $\mu_{\varphi,\alpha}$, we get (see [10, p.163])

$$\begin{aligned} \|RC_{\varphi}M_{\psi}(f)\|^p_{\mathbf{A}^p_\alpha} &= \int_{\mathbb{B}} \Big(\frac{|(R\psi)(\varphi(z))\cdot R\varphi(z)\cdot f(\varphi(z))| + |\psi(\varphi(z))\cdot(Rf)(\varphi(z))\cdot R\varphi(z)|}{|\varphi(z)|}\Big)^p dv_\alpha(z) \\ &= \int_{\mathbb{B}} (|\psi(w)Rf(w)| + |f(w)R\psi(w)|)^p d\mu_{\varphi,\alpha}(w) \\ &\leq \int_{\mathbb{B}} |\psi(w)|^p |Rf(w)|^p d\mu_{\varphi,\alpha}(w) + \int_{\mathbb{B}} |f(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w). \end{aligned}$$

Since $|R\psi|^p d\mu_{\varphi,\alpha}$ is Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$, then

$$\int_{\mathbb{B}} |f(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w) \leq C\|f\|^p_{\mathbf{A}^p_\alpha};$$

On the other hand, for $r > 0$, there exits a constant $C > 0$ such that

$$\mu(D(a,r)) \leq C(1-|a|^2)^{n+1+\alpha+p}$$

holds for $a \in \mathbb{B}$, then by Lemma 2.1,

$$\int_{\mathbb{B}} |\psi(w)|^p |Rf(w)|^p d\mu_{\varphi,\alpha}(w) = \int_{\mathbb{B}} |Rf(w)|^p d\mu(w) \leq C\|f\|^p_{\mathbf{A}^p_\alpha},$$

thus

$$\|RC_{\varphi}M_{\psi}(f)\|^p_{\mathbf{A}^p_\alpha} \leq C\|f\|^p_{\mathbf{A}^p_\alpha},$$

Therefore, $RC_{\varphi}M_{\psi}$ is bounded on $\mathbf{A}^p_\alpha(\mathbb{B})$.

For the converse, assume $RC_{\varphi}M_{\psi}$ is bounded. Then there exists a constant $C > 0$ such that

$$\|RC_{\varphi}M_{\psi}(f)\|^p_{\mathbf{A}^p_\alpha} \leq C\|f\|^p_{\mathbf{A}^p_\alpha}$$

for all $f \in \mathbf{A}^p_\alpha(\mathbb{B})$. Also, there exists a constant $M > 0$ such that $f \in \mathbf{A}^p_\alpha(\mathbb{B})$,

$$\begin{aligned} \|RC_{\varphi}M_{\psi}(f)\|^p_{\mathbf{A}^p_\alpha} &\geq M\int_{\mathbb{B}} |R(\psi f)(w)|^p d\mu_{\varphi,\alpha}(w) \\ &\geq M\int_{\mathbb{B}} |f(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w) - M\int_{\mathbb{B}} |\quad (w)|^p |Rf(w)|^p d\mu_{\varphi,\alpha}(w) \\ &\geq M\int_{\mathbb{B}} |f(w)|^p d\nu(w) - M\int_{\mathbb{B}} |Rf(w)|^p |\quad (w)|^p d\mu_{\varphi,\alpha}(w), \end{aligned}$$

where $d\nu(w) = |R\psi|^p d\mu_{\varphi,\alpha}$. From (1) and lemma 2.1, there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}} |Rf(w)|^p |\psi(w)|^p d\mu_{\varphi,\alpha}(w) \leq C\|f\|^p_{\mathbf{A}^p_\alpha}.$$

then exists a constant $K > 0$ such that

$$\int_{\mathbb{B}} |f(w)|^p d\nu(w) \leq K\|f\|^p_{\mathbf{A}^p_\alpha}.$$

Thus, $d\nu(w) = |R\psi|^p d\mu_{\varphi,\alpha}$ is a Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$.

The proof of the following lemma follows on similar lines as in [1, Proposition 3.11].

Lemma 4.2. Suppose $1 \leq p, q < \infty$. Let $T = RC_\varphi M\psi$. Let φ be a holomorphic mapping defined on $\mathbb{B}$ and $\in \mathbf{H}(\mathbb{B})$ be such that $T : \mathbf{A}^p_\alpha(\mathbb{B}) \to \mathbf{A}^q_\alpha(\mathbb{B})(\alpha > -1)$ is bounded. Then T is compact if and only if whenever $\{f_k\}$ is a bounded sequence in $\mathbf{A}^p_\alpha(\mathbb{B})(\alpha > -1)$ converging to zero uniformly on compact subsets of $\mathbb{B}$, then $\|Tf_k\|_{\mathbf{A}^q_\alpha} \to 0$.

Theorem 4.3. Let $1 \leq p < \infty$, $\alpha > -1$. Let φ be a holomorphic self-map of $\mathbb{B}$ with $\frac{R\varphi}{|\varphi|} \in \mathbf{A}^p_\alpha(\mathbb{B})$ and $\in \mathbf{A}^p_\alpha(\mathbb{B})$ such that $R\psi \in \mathbf{A}^p_\alpha(\mathbb{B})$. Define a finite positive Borel measure $\mu_{\varphi,\alpha}$ on $\mathbb{B}$ by

$$\mu_{\varphi,\alpha}(E) = \int_{\varphi^{-1}(E)} \left(\frac{|R\varphi(z)|}{|\varphi(z)|}\right)^p dv_\alpha(z)$$

for all Borel sets E of $\mathbb{B}$. Let $d\mu(w) = |\psi(\omega)|^p d\mu_{\varphi,\alpha}(w)$. If for every (or some) $r > 0$, there is a constant $C > 0$ such that

$$\lim_{|a| \to 1^-} \frac{\mu(D(a,r))}{(1-|a|^2)^{n+1+\alpha+p}} = 0$$

holds for all $a \in \mathbb{B}$ then $RC_\varphi M\psi$ is compact on $\mathbf{A}^p_\alpha(\mathbb{B})$ if and only if $|R\psi|^p d\mu_{\varphi,\alpha}$ is a vanishing Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$.

Proof. First suppose that $RC_\varphi M\psi$ is compact on $\mathbf{A}^p_\alpha(\mathbb{B})$. Then by using the similar argument as in Theorem 4.1, there exist a constant $C > 0$ such that for $f \in \mathbf{A}^q_\alpha(\mathbb{B})$,

$$\|RC_\varphi M\psi(f)\|^p_{\mathbf{A}^p_\alpha} \geq C\int_{\mathbb{B}} |R(\psi f)(w)|^p d\mu_{\varphi,\alpha}(w).$$

then

$$\begin{aligned}&\int_{\mathbb{B}} |f(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w)\\ \leq\ & C\|RC_\varphi M\psi(f)\|^p_{\mathbf{A}^p_\alpha} + C\int_{\mathbb{B}} |\psi(w)|^p |Rf(w)|^p d\mu_{\varphi,\alpha}(w).\end{aligned}$$

In the above inequality, take $f = k_z(w) \in \mathbf{A}^p_\alpha(\mathbb{B})$, where

$$k_z(w) = \frac{(1-|z|^2)^{\frac{n+1+\alpha}{p}}}{(1-\langle w, z\rangle)^{\frac{2(n+1+\alpha)}{p}}},$$

then

$$\begin{aligned} &\int_{\mathbb{B}} |k_z(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w) \\ \leq\ & C\|RC_\varphi M_\psi(f)\|^p_{\mathbf{A}^p_\alpha} + C\int_{\mathbb{B}} |\psi(w)|^p |Rk_z(w)|^p d\mu_{\varphi,\alpha}(w) \\ =\ & C\|RC_\varphi M_\psi(f)\|^p_{\mathbf{A}^p_\alpha} + C\int_{\mathbb{B}} |Rk_z(w)|^p d\mu. \end{aligned}$$

Since $RC_\varphi M_\psi$ is compact on $\mathbf{A}^p_\alpha$ and the unit vectors k_z tends to 0 uniformly on compact subsets of $\mathbb{B}$ as $|z| \to 0$, by lemma 4.2, $\|RC_\varphi M_\psi(f)\|^p_{\mathbf{A}^p_\alpha} \to 0$ as $|z| \to 0$. On the other hand, sice for every (or some) $r > 0$,

$$\lim_{|a|\to 1^-} \frac{\mu(D(a,r))}{(1-|a|^2)^{n+1+\alpha+p}} = 0,$$

by lemma 2.1,

$$\int_{\mathbb{B}} |Rk_z(w)|^p d\mu \leq \|k_z\|^p_{\mathbf{A}^p_\alpha}.$$

Then, we have

$$\lim_{|z|\to 1^-} \int_{\mathbb{B}} |k_z(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w) = 0.$$

thus, $|R\psi|^p d\mu_{\varphi,\alpha}$ is a vanishing Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$.

Conversely, suppose that $|R\psi|^p d\mu_{\varphi,\alpha}$ is a vanishing Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$. Let $\{f_k\}$ be a norm bounded sequence in $\mathbf{A}^p_\alpha(\mathbb{B})$ $(\alpha > -1)$ such that $\|f_k\|_{\mathbf{A}^p_\alpha} \leq 1$ and $\{f_k\} \to 0$ uniformly on compact subsets of $\mathbb{B}$. Now we prove that $RC_\varphi M$ is compact on $\mathbf{A}^p_\alpha(\mathbb{B})$. By Lemma 4.2, it is enough to show that $\|RC_\varphi M\ (f_k)\|_{\mathbf{A}^p_\alpha} \to 0$ as $k \to \infty$. Using the similar argument as before, we have

$$\|RC_\varphi M_\psi(f_k)\|^p_{\mathbf{A}^p_\alpha} \leq C\int_{\mathbb{B}} |\psi(w)|^p |Rf_k(w)|^p d\mu_{\varphi,\alpha}(w) + C\int_{\mathbb{B}} |f_k(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w).$$

Since $|R\psi|^p d\mu_{\varphi,\alpha}$ is a vanishing Carleson measure on $\mathbf{A}^p_\alpha(\mathbb{B})$, then

$$\lim_{n\to\infty} \int_{\mathbb{B}} |f_k(w)|^p |R\psi(w)|^p d\mu_{\varphi,\alpha}(w) = 0.$$

Using the similar argument as before, we have

$$\lim_{n\to\infty} \int_{\mathbb{B}} |\ \ (w)|^p |Rf_k(w)|^p d\mu_{\varphi,\alpha}(w) = 0.$$

The proof is finished.

REFERENCES

1. C. C. Cowen, B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.
2. J. H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
3. R. A. Hibschweiler, N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, Rock Mountain Journal of Mathematics. Vol. 35, No. 3, 2005.
4. S. Ohno, Products of composition and differention between Hardy spaces, Bull. Austral. Math. Soc. 73, 235-243 (2006).

5. A. K. Sharma, Products of multiplication, composition and differention between weighted Bergman-Nevanlinna and Bloch-tyoe spacers, Turk J Marth 35 (2011), 275-291.
6. A. K. Sharma, S. D. Sharma and S. Kumar, Weighted composition followed by dieeerentiation betwwen Bergman spaces, International Mathematical Forum, 2, 2007, no. 33, 1647-1656. 12
7. S. Kumar, K. J. Singh, Weighted composition operators on weighted Bergman spaces, Extracta Mathematicae. Vol. 22, N´um. 3, 245-256 (2007).
8. H. Li, C. Wang, X. Zhang, Weighted composition followed by dieeerentiation betwwen weighted Bergman space and H∞, Int. Journal of Math. Analysis, Vol. 5, 2011, no. 26, 1267-1272.
9. R. Zhao and K. Zhu, Theory of Bergman spaces in the unit ball of Cn, to appear in Mem. Soc. Math.
10. P. R. Halmos, Measure T heory, Springer-Verlag, New York, 1974.
11. K. Zhu, Spaces of holomorphic functions in the unit ball, New York, Springer-Verlag, 2004.
12. W. Rudin, F unction theory in the unit ball of Cn, New York, Springer-Verlag, 1980. Chao Zhang, Dept. of Math., Guangdong University of Education, Guangzhou, 510310, P. R. China

This page is intentionally left blank