



IMAGE: A MAP OF THE STARS OF THE ORION CONSTELLATION

JournalPreview

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0



Great Britain
Journals Press

JournalPreview

London Journal of Research in Science: Natural & Formal

This document is a pre-published view of London Journal of Research in Science: Natural & Formal Volume 24 Issue 10 and Compilation 1.0. For any minor changes and updations kindly follow your paper's live editing URL given in given in sent email or get in touch with our support team at support@journalspress.com or visit our website to use live chat support. This is a beta document thus order, content or existence of papers may alter in the published eJournal. You are requested to kindly acknowledge and approve your research paper in this JournalPreview within three days.

Journal Content

In this Issue



- i. Journal introduction and copyrights
 - ii. Featured blogs and online content
 - iii. Journal content
 - iv. Editorial Board Members
-

1. An Investigation Into the Order of Integral Powers of Set of All Natural Numbers. **1-14**
 2. On the Well-Posedness for the 3-D Micropolar Fluid System in Critical Fourier-Besov-Morrey Spaces. **15-26**
 3. An Investigation into the Order of Integral Powers of Consecutive Elements of Set of Even and odd Numbers. **27-38**
 4. The Discrete Ordinates and the Riccati Equation Methods in the Estimation of Growth of Solutions of Systems of two Linear first Order Ordinary Differential Equations. **39-64**
-

- v. Great Britain Journals Press Membership



Scan to know paper details and
author's profile

An Investigation of the Order of Integral Powers of Set of all Natural Numbers

Ladan, Umaru Ibrahim, Emmanuel, J. D. Garba & Tanko Ishaya

University of Jos

ABSTRACT

This article established a fact on the order of difference of integral powers of all sets of natural numbers. The analysis was proof by use of established property of difference operator and principle of mathematical induction. The result proved conclusively that “if the elements of an arithmetic progression of set of natural numbers with positive common difference are raised to positive power k , then the k^{th} difference is equal to the product of the common difference raised to power k (d^k) and k factorial ($k!$).

Keywords: finite difference, integral order, positive powers, mathematical induction, arithmetic progression.

Classification: LCC Code: QA241

Language: English



Great Britain
Journals Press

LJP Copyright ID: 925610

Print ISSN: 2631-8490

Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0



An Investigation of the Order of Integral Powers of Set of all Natural Numbers

Ladan, Umaru Ibrahim^α, Emmanuel, J. D. Garba^σ & Tanko Ishaya^ρ

.ABSTRACT

This article established a fact on the order of difference of integral powers of all sets of natural numbers. The analysis was proof by use of established property of difference operator and principle of mathematical induction. The result proved conclusively that “if the elements of an arithmetic progression of set of natural numbers with positive common difference are raised to positive power k, then the kth difference is equal to the product of the common difference raised to power k (d^K) and k factorial (k!).

Keywords: finite difference, integral order, positive powers, mathematical induction, arithmetic progression.

Author α: Department of Computer Science, Faculty of Natural Sciences, University of Jos.

σ: Department of Mathematics, Faculty of Natural Sciences, University of Jos.

ρ: Department of Computer science, Faculty of Natural Sciences, University of Jos.

I. INTRODUCTION

A set of positive integers is a set of natural numbers. A set is a collection of well-defined elements. A set of natural number is a sequence, defined with respect to a constant value called a common difference (d). The terms of a sequence are defined with respect to three parameters, the first term (a), the numbers of terms in the sequence (n) and the common difference (d). The sequence of natural numbers expressed mathematically as:

$$T_n = a + (n - 1) d \dots\dots\dots I$$

$$\text{For all } a \geq 1, d > 1, n \in (0, \infty)$$

Any sequence generated with the above formular is called an **arithmetic progression**.

A perfect power is a rational root chase.^[1] An integral perfect power is the irrational root that is an integer. A finite difference is a mathematical expression of the form, $f(x + b) - f(x + a)$ and a forward difference is of the form $\Delta h[f](x) = f(x + h) - f(x)$ ^[2]

This article investigates all Arithmetic Progression of set of natural numbers with positive common difference. Further investigation reveals that there is a significant relationship between the difference of powers of set of natural numbers and factorial, which is represented symbolically as:

$$\Delta^k [a + (n - 1)d]^k = d^k K! \dots\dots\dots II$$

$$\text{For all } a \geq 1, d \geq 1, k > 1 \text{ and}$$

For some $n \in (0, \infty)$

The implication of equation (II) is on the coefficient of the k th factorial, which shows the value of k is the power of the common difference and at the same time is the power of the sequence under consideration for which the k th difference of the set is equal to the product of the common difference raised to power k and $k!$.

An examination of any set of natural numbers having common difference greater than 2 e.g. $\{2, 5, 8, 11, 14, 17, 20, 23, 26, 29\}$ reveals that the pattern is sustained (See Table I and Appendix A).

The set of perfect squares of the set is $\{4, 25, 64, 121, 196, 289, 400, 529, 676, 841\}$. The actual set of the first difference is $\{21, 39, 57, 75, 93, 111, 129, 147, 165\}$. Furthermore, the set of the second difference is $\{18, 18, 18, 18, 18, 18, 18, 18\} = \{18\}$, a singleton set. Clearly, the set of differences of the above second difference set is $\{0\}$. The emerging pattern motivates the investigation of whether the pattern will persist for all set of natural numbers with positive common difference $d > 2$. The set of the second difference of the set of perfect squares clearly shows that the element of the singleton set $18 = 3^2(2!)$.

This shows that for $d = 3, k = 2$.

$$\begin{aligned} \Delta^2 [a + (n - 1)d]^2 &= d^k \cdot k!, \text{ where} \\ a = 2, d = 3, k = 2, \text{ and } n \in (0,9) \\ \Rightarrow \Delta^2 [a + (n - 1)d]^2 &= 18 = 3^2(2!) \end{aligned}$$

In their contribution in Exponential Diophantine Equations, Shorey and Tijdeman investigated perfect powers at integral values of a polynomial with rational integer coefficients and obtain in particular the following. Let $f(x)$ be a polynomial with rational coefficients and with at least two simple rational zeros.

Suppose $b \neq 0, m \geq 0, x$ and y with $|y| > 1$ are rational integers. Then the equation $f(x) = by^m$ implies that m is bounded by a computable number depending only on b and f . Also in their contribution Ladan, Tanko, Aliyu, Ahmad and Kabiru^[4], they investigated integral powers of polynomials with Binomial coefficients. The result of their investigation shows that “the disposition of powers of polynomials with binomial coefficients generates even positive factorial $(2k)!$ ”

Also in a research work by Ladan, Aliyu, Tanko, Ahmad and Kabiru^[5], on the location of points on the plane and the order of disposition of sum of powers of cardinal coordinates. The result of their work proved conclusively that “the sum of the powers of cardinal points is equal to the coefficients of the Binomial expansion with respect to the Pascal triangle pattern and entries”. In their contribution in the difference of perfect powers of integers, Ladan, Ukwu

and Apine^[6] investigated order of integral perfect powers and proved that “if any number of consecutive integers are raised to a positive integral power k , then the k^{th} difference is equal to $k!$ Based on the generalization of the theorem in this article, it shows that Ladan, Ukwu and Apine work has common difference $d = 1$, for all $k > 1$.

Similarly, in a research conducted by Ladan, Emmanuel and Tanko^[7], investigated order of product of perfect powers and proved conclusively that if any number of consecutive integer are raised to a positive power k , then the $(2k)^{\text{th}}$ difference of the product of the k^{th} power of two consecutive integers is equal to $(2k)!$. This established a relation between difference of powers of natural numbers and factorial. For more definitions, see [8, 9, 10].

There was no discussion on the relationship between the difference of powers of set of all natural numbers and factorial with respect to the common difference which is pertinent to this article. Review of literature shows that no such investigation has been undertaken. Thus, this article adds to the existing body of knowledge.

II. METHODS

2.1 Preliminary Definitions

In what follows, the difference of finite order will be defined.

2.1.1 Differences of Order One (1)

Given a sequence $\{f_j\}_1^\infty$, defined the difference of order one at j with respect to the sequence by:

$$\Delta(f_j) = f_{j+1} - f_j, \text{ for every } j$$

2.1.2 Higher Order Differences

Higher order differences can be defined recursively by:

$$\Delta^k(f_j) = \Delta(\Delta^{k-1}(f_j)) = \Delta^{k-1}(\Delta(f_j)) = \Delta^{k-1}[f_{j+1} - f_j] \text{ for } \geq 2$$

III. RESULTS AND DISCUSSION

3.1 Preliminary Theorem

Suppose $f_j = j$ for all j belong to set of natural numbers.

Then:

- (i) Δf_j^2 is natural number
- (ii) $\Delta^2(f_j^2) = 18 = 3^2 \cdot (2!)$
- (iii) $\Delta^k(f_j^2) = 0, k \in \{3, 4, \dots\}$

Proof

Let $j = \{2, 5, 8, 11, 14, 17, 20, 23, 26, 29\}$

Then Table I below yields value of $\Delta^k(f_j^p)$ for selected values of k and p in the set $\{0, 1, 2, 3\}$ and $\{1, 2\}$ respectively, when $\Delta^0(f_j^p) = f_j^p$

Table I: Difference Order Table for Selected Elements of Set of Natural Numbers

| f_j | f_j^2 | $\Delta(f_j^2)$ | $\Delta^2(f_j^2)$ | $\Delta^3(f_j^2)$ |
|-------|---------|-----------------|-------------------|-------------------|
| 2 | 4 | 21 | 18 | 0 |
| 5 | 25 | 39 | 18 | 0 |
| 8 | 64 | 57 | 18 | 0 |
| 11 | 121 | 75 | 18 | 0 |
| 14 | 196 | 93 | 18 | 0 |
| 17 | 289 | 111 | 18 | 0 |
| 20 | 400 | 129 | 18 | 0 |
| 23 | 529 | 147 | 18 | 0 |
| 26 | 676 | 165 | 18 | |
| 29 | 841 | | | |

It is clear that $\Delta^k(f_j^k) = d^k k!$ Where $d = 3, k = 2$, and $j \in \{2, 5, \dots, 26, 29\}$.

That is $\Delta^2(f_j^2) = 3^2(2!) = 18$

Obviously, the theorem is valid for $j \in \{2, 5, 8, 11, 14, 17, 20, 23, 26, 29\}$ as observed from column 4. The proofs of (i), (ii) and (iii) are direct.

- (i) $\Delta(f_j^2) = f_{j+1}^2 - f_j^2 = (j + 1)^2 - j^2 = 2j + 1$, which is natural number for all j , proving (i).
- (ii) $\Delta(\Delta f_j^2) = 3^2 \Delta(2j + 1) = 3^2(\Delta 2j + \Delta 1) = 3^2(2\Delta j) = 3^2[2(j + 1 - j)] = 3^2 2[1] = 3^2(2) = 3^2 \cdot 2 = 18$, for $d = 3$, proving (ii).

The principle of mathematical induction is needed in the proof of (iii).

For $k = 3, \Delta^3(f_j^2) = \Delta(\Delta^2(f_j^2)) = \Delta(18) = 18 - 18 = 0$

Consequently $\Delta^k(f_j^2) = \Delta^{k-3}(\Delta^3(f_j^2)) = \Delta^{k-3}(0) = 0$

For all $k \geq 4$, proving $\Delta^k(f_j^2) = 0$, for all positive integer $k \geq 3$. This established the proof of (iii).

3.2 Main Theorem

Let $f_j = j$

Let $\{j_n\}_1^\infty$ be a sequence defined by $j_n = a + (n - 1)d$, for all $n \geq 1$, $d \geq 1$ and $a \geq 1$.

Let $[j_n]^k = [a + (n - 1)d]^k$, for $k = \{1, 2, 3, \dots\}$.

We shall consider the case for the power of $k = \{1, 2, 3\}$ to establish the proof of the theorem.

Case (1), $k = 1$, $n = \{1, 2, 3, \dots\}$

$[j_n]^k = [a + (n - 1)d]^k$, $k = 1$ are constant.

$\Rightarrow [j_n]^1 = [a + (n - 1)d]^1$ is the sequence.

The first difference is given as:

$$\Delta^1[j_n]^1 = \Delta^1[a + (n - 1)d]^1,$$

$$\begin{aligned}\Delta[j_n] &= \Delta(a + dn - d) \\ &= \Delta a + \Delta dn - \Delta d \\ &= \Delta dn \\ &= d[(n + 1) - n] \\ &= d(1) \\ &= d \\ &= d^1(1!)\end{aligned}$$

Therefore, for $k = 1$

$$\Delta^1[j_n]^1 = \Delta^1[a + (n - 1)d]^1 = d^1(1!)$$

Case (2), $k = 2$, $n = \{1, 2, 3, \dots\}$

$[j_n]^2 = [a + (n - 1)d]^2$ is set of perfect squares.

$$\Delta^2[j_n]^2 = \Delta^2[a + (n - 1)d]^2,$$

$$\begin{aligned}\Delta[\Delta j_n^2] &= \Delta^2[a + dn - d] \\ &= \Delta^2[(a + dn - d)(a + dn - d)] \\ &= \Delta^2[a + adn - ad + adn + d^2n^2 - d^2n - ad - d^2n + d^2] \\ &= \Delta^2[a^2 + 2adn - 2ad + d^2n^2 - 2d^2n + d^2]\end{aligned}$$

$$\begin{aligned}
 &= \Delta[\Delta(a^2 + 2adn - 2ad + d^2n^2 - 2d^2n + d^2)] \\
 &= \Delta[\Delta a^2 + 2ad\Delta n - \Delta 2ad + d^2\Delta(n^2) - 2d^2\Delta(n) - \Delta d^2] \\
 &= \Delta[2ad((n + 1) - (n) + d^2((n + 1)^2 - (n)^2) - 2d^2((n + 1) - (n))] \\
 &= \Delta[2ad(1) - d^2((n^2 + 2n - 1) - (n)^2 - 2d^2(1))] \\
 &= \Delta[2ad + d^2(2n - 1) - 2d^2] \\
 &= \Delta 2ad + \Delta 2d^2n - \Delta 3d^2 \\
 &= \Delta 2d^2n \\
 &= 2d^2\Delta(n) \\
 &= 2d^2(n + 1 - n) \\
 &= 2d^2(1) = d^2 2!
 \end{aligned}$$

Therefore,

$$\Delta^2[j_n]^2 = \Delta^2[a + (n - 1)d]^2 = d^2(2!), \text{ satisfied.}$$

Case (3), k = 3, n = {1, 2, 3, ...}

$$\begin{aligned}
 \Delta^3[j_n]^3 &= \Delta^3[a + (n - 1)d]^3 \\
 &= \Delta^3[(a + dn - d)^3] \\
 &= \Delta^3[(a + dn - d)(a + dn - d)^2] \\
 &= \Delta^3[(a + dn - d)(a^2 + 2adn - 2ad - 2d^2n + d^2n^2 + d^2)] \\
 &= \Delta^3[a^3 + 2a^2dn - 2a^2d - 2ad^2n + ad^2n^2 + ad^2 + a^2dn + 2ad^2n^2 - 2ad^2n^2 - \\
 &\quad 2d^3n^2 + d^3n^3 + d^3n - a^2d - 2ad^2n + 2ad^2 + 2d^3n - d^3n^2 - d^3] \\
 &= \Delta^3[a^3 + 3a^2dn - 3a^2d - 6ad^2n + 3ad^2n^2 + 3ad^2 - 3d^3n^2 + d^3n^3 + 3d^3n - d^3] \\
 &= \Delta^2[\Delta(a^3 + 3a^2dn - 3a^2d - 6ad^2n + 3ad^2n^2 + 3ad^2 - 3d^3n^2 + d^3n^3 + 3d^3n - \\
 &\quad d^3)] \\
 &= \Delta^2[\Delta(a^3 + \Delta 3a^2dn - \Delta 3a^2d - \Delta 6ad^2n + \Delta ad^2n^2 + \Delta 3ad^2 - \Delta 3d^3n^2 + \Delta d^3n^3 + \\
 &\quad \Delta 3d^3n - \Delta d^3)] \\
 &= \Delta^2[3a^2d\Delta n - 6ad^2\Delta n + ad^2\Delta n^2 + 3d^3\Delta n^2 + d^3\Delta n^3 + 3d^3\Delta n] \\
 &= \Delta^2[3a^2d(n + 1 - n) - 6ad^2(n + 1 - n) + ad^2((n + 1)^2 - n^2) - 3d^3((n + 1)^2 - \\
 &\quad n^2) + d^3((n + 1)^3 - n^3) + 3d^3(n + 1 - n)] \\
 &= \Delta^2[3a^2d - 6ad^2(n^2 + 2n + 1 - n^2) - 3d^3n^2 + 2n + 1 - n^2 + d^3([n + 1](n^2 + \\
 &\quad 2n + 1 - n^3 + 3d^3)] \\
 &= \Delta^2[3a^2d - 6ad^2 + ad^2(2n + 1) - 3d^3(2n + 1) + d^3(n^3 + 3n^2 + 3n + 1) - n^3 + \\
 &\quad 3d^3]
 \end{aligned}$$

$$\begin{aligned}
 &= \Delta^2[3a^2d - 5ad^2 + 2ad^2n - 3d^3n - 2d^3 + 3d^3n^2 + 3d^3] \\
 &= \Delta[\Delta(3a^2d - 5ad^2 + 2ad^2n - 3d^3n - 2d^3 + 3d^3n^2 + d^3)] \\
 &= \Delta[\Delta(3a^2d - 5ad^2 + 2ad^2n - 3d^3n + \Delta(3d^3n^2 + d^3))] \\
 &= \Delta[2ad^2\Delta n - 3d^3\Delta n + 3d^3\Delta n^2] \\
 &= \Delta[2ad^2(n + 1 - n) - 3d^3(n + 1 - n) + 3d^3(n^2 + 2n + 1 - n^2)] \\
 &= \Delta[2ad^3 - 3d^3 + 3d^3(2n + 1)] \\
 &= \Delta[2ad^3 - 3d^3 + 6d^2 + 3d^3] \\
 &= \Delta(2ad^3 - \Delta(3d^3) + \Delta(6d^2 + 3d^3)) \\
 &= \Delta(6d^3n) \\
 &= 6d^3\Delta n \\
 &= 6d^3 \\
 &= d^3(6) \\
 &= d^33!
 \end{aligned}$$

Therefore,

$$\Delta^3[j_n]^3 = \Delta^3[a + (n - 1)d]^3 = d^3(3!)$$

Thus, the proof of the theorem is established for the values integral power of $k = \{1, 2, 3\}$. This clearly reveals that the pattern is sustained for the values of k belong to natural numbers for all $d > 2$ and $n \in (1, \infty)$.

Assume the validity of the theorem of $k = \{4, 5, \dots m\}$ for some natural number $m \geq 4$.

Inductively, the theorem holds for $k = 1$, see case (1) $\Delta[j_n] = \Delta[a + (n - 1)d] = d \cdot 1!$

The truth of theorem for $k = 1$, implies is true for all positive $k > 1$, that is $\Delta^k[j_n]^k = \Delta^k[a + (n - 1)d]^k = d^k \cdot k!$, for some $k > 1$. (See Case 2 and 3).

Thus by induction hypothesis, the truth of the theorem for k implies the validity of the theorem for $k + 1$.

Therefore,
$$\Delta^{k+1}[j_n]^{k+1} = \Delta^{k+1}[a + (n - 1)d]^{k+1}$$

$$\Delta^{k+1}[a + (n - 1)d]^{k+1} = d^{k+1}(k + 1)!$$

This completes the proof, that is the pattern is sustained for all powers of set of natural numbers, for all $k \geq 1$ and $d \geq 1, a \geq 1, n \in \{1, 2, 3, \dots m\}, m \geq 1$.

The theorem is established.

3.3 Corollary

$\Delta^k [j_n]^k = d^k \cdot k!$ for every arithmetic progression of set of natural numbers j_n and for any positive integer k , for all $d \geq 2$. In other words, for any positive integer k , the k^{th} order difference of the k^{th} power of a sequence of natural numbers is equal to $d^k \cdot k!$ where d is the common difference, $d \geq 1$.

By implication, the theorem states that “if the elements of an arithmetic progression of the set of natural numbers with positive common difference are raised to a positive power k , then the k^{th} difference is equal to the product of the common difference and k factorial $[d^k \cdot k!]$.”

3.4 Remarks

The following strong relationship exist between the difference operator Δ and D operator (differential operator).

$$D^k(K^p) = 0 \text{ if } k > p$$

$$D^k x^k = k!$$

The coefficient of x^{p-k}

$$D^k(K^p) = \frac{p!}{(p-k)!} x^{p-k} \text{ for } k \geq 2, k < p \text{ is even.}$$

IV. CONCLUSION

This article established the structures of finite difference orders with respect to the powers of set of natural numbers. The result reveals a startling relationship between the common difference of the sequence and $k!$, as reflected in the theorem.

Competing Interests: Authors have declared that no competing interest exist.

REFERENCES

1. Chase, P.E. The elements of arithmetic... in which decimal and integral arithmetic are combined and taught inductively. U. Hunt and Son; 1984.
2. Finite difference, 2015. Retrieved October 2015. Available at: <https://www.en.wikipedia.org/wiki/finite-difference>
3. Cambridge Tracts in Mathematics, 87. Exponential Diophantine Equations. Cambridge University Press 1986, 169-183.
4. Ladan, Tanko, Aliyu, Ahmad and Kabiru. An Exposition of Integral Powers of Polynomials with Binomial Coefficient. Ladan et al./KJSET: Vol 01, No. 02, (Dec 2022) 18-25, ISSN: 1958-0641. <https://www.kjset.ku.ac.ug/>
5. Ladan, Aliyu, Tanko, Ahmad and Kabiru. Location of Points on Plane and the Order of Disposition of Sum of Powers of Cardinal Coordinates. Ladan et al./KJSET, Vol. 01, No. 02 (Dec 2022) 18-25, ISSN: 1958-0641. <https://www.kjset.ku.ac.ug/>

6. Ladan Umaru Ibrahim, Ukwu Chukwunenye and Apine Elijah. An investigation into the Order of Integral Perfect Powers. Ibrahim et al. AJOMCOR, 10(1) 12:18:2016.
7. Ibrahim, L. U., Garba, E. J. D., & Ishaya, T. (2023). An investigation into the Order of Difference of Product of Consecutive Integral Perfect Powers. Journal of Advances in Mathematics and Computer Science, 38 (11), 47-56. <https://doi.org/10.9734/james/2023/V38i111844>
8. Saylor.org Academy Sequence defined by a Recursive Formula, using Factorial Notation.learn.Saylor.org>mod>book>view.
9. Robert Blitzer College Algebra Second Edition Miami-Dade Community College. College Algebra/Robert Blitzer, 2nd ed. PCM ISBN 0-13-08728-61, I Algebra I. Title QA15.2.B5842001.
10. Perfect Powers: 2015 Retrieved November 23, 2015. Available at: <https://en.wikipedia.org/wiki/perfect-power>
11. Andreescu, T. Andrica D. Number theory. Birkhauser Boston, a part of Springer Science + Business Media, LLC; 2009. DOI: 10:1007/b//856-2.
12. Tarlok Shorey and Robert Tijdeman: Perfect Powers in Products of terms in arithmetic progression III. January 1992, Compositio Mathematica 82 (2) DOI. 10-1090/COMM/210
13. Shorey T. N. Perfect Powers in Values of Certain Polynomials at Integer Points Maths. Proc. Camb. Phil. Soc. 1986; 99 195 – 207.

APPENDIX A

Table 1: d = 3, k = 2

| j_n | j_n^2 | Δj_n^2 | $\Delta^2 j_n^2$ | $\Delta^3 j_n^2$ |
|-------|---------|----------------|------------------|------------------|
| 2 | 4 | 21 | 18 | 0 |
| 5 | 25 | 39 | 18 | 0 |
| 8 | 64 | 57 | 18 | 0 |
| 11 | 121 | 75 | 18 | 0 |
| 14 | 196 | 93 | 18 | 0 |
| 17 | 289 | 111 | 18 | 0 |
| 20 | 400 | 129 | 18 | 0 |
| 23 | 529 | 147 | 18 | 0 |
| 26 | 676 | 165 | 18 | |
| 29 | 841 | | | |

We have that: $\Delta^2 j_n^2 = 18 = 3^2(2!)$

Table 2: d = 3, k = 2

| j_n | j_n^3 | Δj_n^3 | $\Delta^2 j_n^3$ | $\Delta^3 j_n^3$ | $\Delta^4 j_n^3$ |
|-------|---------|----------------|------------------|------------------|------------------|
| 2 | 8 | 117 | 270 | 162 | 0 |
| 5 | 125 | 387 | 432 | 162 | 0 |
| 8 | 512 | 819 | 594 | 162 | 0 |
| 11 | 1331 | 1413 | 756 | 162 | 0 |
| 14 | 2744 | 2169 | 918 | 162 | |
| 17 | 4913 | 3087 | 1080 | | |
| 20 | 8000 | 4167 | | | |
| 23 | 12167 | | | | |

We have that: $\Delta^3 j_n^3 = 162 = 3^3(3!)$

Table 3: $d = 3, k = 2$

| j_n | j_n^2 | Δj_n^2 | $\Delta^2 j_n^2$ | $\Delta^3 j_n^2$ |
|-------|---------|----------------|------------------|------------------|
| 1 | 1 | 15 | 18 | 0 |
| 4 | 16 | 33 | 18 | 0 |
| 7 | 49 | 51 | 18 | 0 |
| 10 | 100 | 69 | 18 | 0 |
| 13 | 169 | 87 | 18 | |
| 16 | 256 | 105 | | |
| 19 | 361 | | | |

We have that: $\Delta^2 j_n^2 = 18 = 3^2(2!)$

Table 4: $d = 3, k = 4$

| j_n | j_n^4 | Δj_n^4 | $\Delta^2 j_n^4$ | $\Delta^3 j_n^4$ | $\Delta^4 j_n^4$ | $\Delta^5 j_n^4$ |
|-------|---------|----------------|------------------|------------------|------------------|------------------|
| 2 | 16 | 609 | 2862 | 4212 | 1944 | 0 |
| 5 | 625 | 3471 | 7074 | 6156 | 1944 | 0 |
| 8 | 4096 | 10545 | 13230 | 8100 | 1944 | 0 |
| 11 | 14641 | 23775 | 21330 | 10044 | 1944 | 0 |
| 14 | 38416 | 45105 | 31374 | 11988 | 1944 | |
| 17 | 83521 | 76479 | 43362 | 13932 | 1944 | |
| 20 | 16000 | 119841 | 57294 | 15876 | | |
| 23 | 279841 | 177135 | 73170 | | | |
| 26 | 456976 | 250305 | | | | |
| 29 | 707281 | | | | | |

We have that: $\Delta^4 j_n^4 = 1944 = 3^4(4!)$

Table 5: $d = 3, k = 5$

| j_n | j_n^5 | Δj_n^5 | $\Delta^2 j_n^5$ | $\Delta^3 j_n^5$ | $\Delta^4 j_n^5$ | $\Delta^5 j_n^5$ | $\Delta^6 j_n^5$ |
|-------|----------|----------------|------------------|------------------|------------------|------------------|------------------|
| 2 | 32 | 3093 | 26550 | 72090 | 77760 | 29160 | 0 |
| 5 | 3125 | 29643 | 98640 | 149850 | 106920 | 29160 | 0 |
| 8 | 32768 | 128283 | 248490 | 256770 | 136080 | 29160 | 0 |
| 11 | 161051 | 376773 | 505260 | 392850 | 165240 | 29160 | 0 |
| 14 | 537824 | 882033 | 898110 | 558090 | 194400 | 29160 | |
| 17 | 1419857 | 1780143 | 1456200 | 752490 | 223560 | | |
| 20 | 3200000 | 3236343 | 2208690 | 976050 | | | |
| 23 | 6436343 | 5445033 | 3184740 | | | | |
| 26 | 11881376 | 8629773 | | | | | |
| 29 | 20511149 | | | | | | |

We have that: $\Delta^5 j_n^5 = 29160 = 3^5(5!)$

Table 6: $d = 4, k = 2$

| j_n | j_n^2 | Δj_n^2 | $\Delta^2 j_n^2$ | $\Delta^3 j_n^2$ |
|-------|---------|----------------|------------------|------------------|
| 2 | 4 | 32 | 32 | 0 |
| 6 | 36 | 64 | 32 | 0 |
| 10 | 100 | 96 | 32 | 0 |
| 14 | 196 | 128 | 32 | |
| 18 | 324 | 160 | | |
| 22 | 484 | | | |

We have that: $\Delta^2 j_n^2 = 32 = 4^2(2!)$

Table 7: $d = 4, k = 3$

| j_n | j_n^3 | Δj_n^3 | $\Delta^2 j_n^3$ | $\Delta^3 j_n^3$ | $\Delta^4 j_n^3$ |
|-------|---------|----------------|------------------|------------------|------------------|
| 2 | 8 | 208 | 576 | 384 | 0 |
| 6 | 216 | 784 | 960 | 384 | 0 |
| 10 | 1000 | 1744 | 1344 | 384 | 0 |
| 14 | 2744 | 3088 | 1728 | 384 | 0 |
| 18 | 5832 | 4816 | 2112 | 384 | 0 |
| 22 | 10648 | 6928 | 2496 | 384 | |
| 26 | 17576 | 9424 | 2880 | | |
| 30 | 27000 | 12304 | | | |
| 38 | 55136 | | | | |

We have that: $\Delta^4 j_n^3 = 384 = 4^3(3!)$

Table 8: $d = 4, k = 4$

| j_n | j_n^4 | Δj_n^4 | $\Delta^2 j_n^4$ | $\Delta^3 j_n^4$ | $\Delta^4 j_n^4$ | $\Delta^5 j_n^4$ |
|-------|---------|----------------|------------------|------------------|------------------|------------------|
| 2 | 16 | 1280 | 7424 | 12288 | 6144 | 0 |
| 6 | 1296 | 8704 | 19712 | 18432 | 6144 | 0 |
| 10 | 10000 | 28416 | 38144 | 24576 | 6144 | 0 |
| 14 | 38416 | 66560 | 62720 | 30720 | 6144 | 0 |
| 18 | 104976 | 129280 | 93440 | 36864 | 6144 | 0 |
| 22 | 234256 | 222720 | 130304 | 43008 | 6144 | |
| 26 | 456976 | 353024 | 173312 | 49152 | | |
| 30 | 810000 | 526336 | 222464 | | | |
| 34 | 1336336 | 748800 | | | | |
| 38 | 2085136 | | | | | |

We have that: $\Delta^4 j_n^4 = 6144 = 4^4(4!)$

Table 9: $d = 4, k = 5$

| j_n | j_n^5 | Δj_n^5 | $\Delta^2 j_n^5$ | $\Delta^3 j_n^5$ | $\Delta^4 j_n^5$ | $\Delta^5 j_n^5$ | $\Delta^6 j_n^5$ |
|-------|----------|----------------|------------------|------------------|------------------|------------------|------------------|
| 1 | 1 | 3124 | 52800 | 203520 | 276480 | 122880 | 0 |
| 5 | 3125 | 55924 | 256320 | 480000 | 399360 | 122880 | 0 |
| 9 | 59049 | 312244 | 736320 | 879360 | 522240 | 122880 | 0 |
| 13 | 371293 | 1048654 | 1615680 | 1401600 | 645120 | 122880 | |
| 17 | 1419857 | 2664244 | 3017280 | 2046720 | 768000 | | |
| 21 | 4084101 | 5681524 | 5064000 | 2814720 | | | |
| 25 | 9765625 | 10745524 | 7878720 | | | | |
| 29 | 20511149 | 18624244 | | | | | |
| 33 | 39135393 | | | | | | |

We have that: $\Delta^5 j_n^5 = 122880 = 4^5(5!)$

Table 10: $d = 7, k = 2$

| j_n | j_n^5 | Δj_n^5 | $\Delta^2 j_n^5$ |
|-------|---------|----------------|------------------|
| 2 | 4 | 77 | 98 |
| 9 | 81 | 175 | 98 |
| 16 | 256 | 273 | 98 |
| 23 | 527 | 271 | |
| 30 | 900 | | |

We have that: $\Delta^2 j_n^2 = 98 = 7^2(2!)$

Table 11: $d = 7, k = 3$

| j_n | j_n^3 | Δj_n^3 | $\Delta^2 j_n^3$ | $\Delta^3 j_n^3$ | $\Delta^4 j_n^3$ |
|-------|---------|----------------|------------------|------------------|------------------|
| 2 | 8 | 721 | 2646 | 2058 | 0 |
| 9 | 729 | 3367 | 4704 | 2058 | 0 |
| 16 | 4096 | 8071 | 6762 | 2058 | 0 |
| 23 | 12167 | 14833 | 8820 | 2058 | 0 |
| 30 | 27000 | 23653 | 10878 | 2058 | |
| 37 | 50653 | 34531 | 12936 | | |
| 44 | 85184 | 47467 | | | |
| 51 | 132651 | | | | |

We have that: $\Delta^3 j_n^3 = 2058 = 7^3(3!)$

Table 12: $d = 7, k = 4$

| j_n | j_n^4 | Δj_n^4 | $\Delta^2 j_n^4$ | $\Delta^3 j_n^4$ | $\Delta^4 j_n^4$ | $\Delta^5 j_n^4$ |
|-------|---------|----------------|------------------|------------------|------------------|------------------|
| 2 | 16 | 6545 | 52430 | 102900 | 57624 | 0 |
| 9 | 6561 | 58975 | 155330 | 160524 | 57624 | 0 |
| 16 | 65536 | 214305 | 315854 | 218148 | 57624 | 0 |
| 23 | 279841 | 530159 | 534002 | 275772 | 57624 | 0 |
| 30 | 810000 | 1064161 | 809774 | 333396 | | |
| 37 | 1874161 | 1873935 | 1143170 | | | |
| 44 | 3748096 | 3017105 | | | | |
| 51 | 6765201 | | | | | |

We have that: $\Delta^4 j_n^4 = 57624 = 7^4(4!)$

Table 13: $d = 19, k = 3$

| j_n | j_n^3 | Δj_n^3 | $\Delta^2 j_n^3$ | $\Delta^3 j_n^3$ | $\Delta^4 j_n^3$ |
|-------|---------|----------------|------------------|------------------|------------------|
| 2 | 8 | 9253 | 45486 | 41154 | 0 |
| 21 | 9261 | 54739 | 86640 | 41154 | 0 |
| 40 | 64000 | 141379 | 127794 | 41154 | 0 |
| 59 | 205379 | 269173 | 168948 | 41154 | 0 |
| 78 | 474552 | 438121 | 210102 | 41154 | 0 |
| 97 | 912673 | 648223 | 251256 | 41154 | 0 |
| 116 | 1560896 | 899479 | 292410 | 41154 | |
| 135 | 2460375 | 1191889 | 333564 | | |
| 154 | 3652264 | 1525453 | | | |
| 173 | 5177717 | | | | |

We have that: $\Delta^3 j_n^3 = 41154 = 19^3(3!)$

This page is intentionally left blank



Scan to know paper details and
author's profile

On the Well-Posedness for the 3-D Micropolar Fluid System in Critical Fourier-Besov-Morrey Spaces

Fatima Ouidirne, Mohamed Oukessou & Jalila El ghordaf

Sultan Moulay Slimane University

ABSTRACT

In the present paper, we study the Cauchy problem of the incompressible micropolar fluid system in \mathbb{R}^3 . We show that this problem is locally well-posed in Fourier-Besov-Morrey spaces $\mathcal{FN}_{1,\lambda,q}$ for $1 \leq q \leq \infty$, and is globally well-posed in these spaces with small initial data.

Keywords: 3-D micropolar fluid system, Fourier-Besov-Morrey spaces, well-posedness.

Classification: MSC Code: 35Q35

Language: English



Great Britain
Journals Press

LJP Copyright ID: 925612

Print ISSN: 2631-8490

Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0



On the Well-Posedness for the 3-D Micropolar Fluid System in Critical Fourier-Besov-Morrey Spaces

Fatima Ouidirne^a, Mohamed Oukessou^o & Jalila El ghordaf^p

ABSTRACT

In the present paper, we study the Cauchy problem of the incompressible micropolar fluid system in \mathbb{R}^3 . We show that this problem is locally well-posed in Fourier-Besov-Morrey spaces $\mathcal{F}\mathcal{N}_{1,\lambda,q}$ for $1 \leq q \leq \infty$, and is globally well-posed in these spaces with small initial data.

Keywords: 3-D micropolar fluid system, Fourier-Besov-Morrey spaces, well-posedness.

Author α σ ρ : Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay Slimane University, Morocco.

I. INTRODUCTION

In this paper, we are interested in the following initial value problem for the system of partial differential equations describing the motion of incompressible micropolar fluid:

$$\begin{cases} \partial_t u - (\chi + \nu)\Delta u + u \cdot \nabla u + \nabla \pi - 2\chi \nabla \times \omega = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \partial_t \omega - \mu \Delta \omega + u \cdot \nabla \omega + 4\chi \omega - \kappa \nabla \operatorname{div} \omega - 2\chi \nabla \times u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ (u, \omega)|_{t=0} = (u_0, \omega_0) & \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

where $u = u(x, t)$, $\pi = \pi(x, t)$ and $\omega = \omega(x, t)$ are unknown functions representing the linear velocity field, the pressure field of the fluid and the micro-rotation velocity field, respectively. κ, μ, ν and χ are positive constants reflecting various viscosity of the fluid. Throughout this paper we only consider the situation with $\kappa = \mu = 1$ and $\chi = \nu = 1/2$.

Theory of micropolar fluid was proposed by Eringen [8] in 1996, his idea allows us to consider several physical phenomena which cannot be treated by the classical Navier-Stokes system for the viscous incompressible fluid, then the problem (1) was presented as a necessary modification to the traditional Navier-Stokes equations in order to better characterize the motion of real-world fluids consisting of rigid but randomly oriented particles (such as blood) by examining the influence of micro-rotation of the particles suspended in the fluid.

There are several results on the weak and strong solvency of the micropolar fluid system and some related topics. The weak solution of (1) was firstly considered by Galdi and Rionero [11]. The existence theorem of the micropolar fluid system with sufficiently regular initial data has been showed by Lukaszewicz [15]. Inoue et al. [12] proved similar result for the magneto-micropolar fluid system. Many authors obtained the well-posedness of the problem (1) in various function spaces.

For instance in the Besov spaces $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $p \in [1, 6)$ and $q = \infty$, Chen and Miao [5] obtained global well-posedness of the problem (1) for small initial data. Zhu and Zhao [21] proved that the Cauchy problem (1) is locally well-posed in the Fourier-Besov spaces $\mathcal{F}\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $1 < p \leq \infty$ and $1 \leq q < \infty$ and globally well posed in these spaces with small initial data. Recently, Weipeng Zhu [22] considered a critical case $p = 1$ and showed that this problem is locally well-posed in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $1 \leq q \leq 2$, and is globally well-posed in these spaces with small initial data. Also, Zhu proved the ill-posedness of (1) in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $2 < q \leq \infty$. In addition, by using a similar argument Zhu established the ill-posedness of (1) in Besov spaces $\dot{B}_{\infty,q}^{-1}$ with $2 < q \leq \infty$. The well-posedness of a more general model than (1) is established by Ferreira and Villamizar-Roa [9] in pseudo-measure spaces. For the other studies of the problem (1), we refer to the monographs [6, 16, 17, 20].

We remark that if $\chi = 0$ and $\omega = 0$, then we have the classical Navier-Stokes equations:

$$\begin{cases} u_t - \mu \Delta u + (u \cdot \nabla)u + \nabla p = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3. \end{cases}$$

The local and global well-posedness of the classical Navier-Stokes equations have been established by a lot of researches in various function spaces, we refer to [13, 14] and references cited therein.

In the present paper, we show that the problem (1) is locally well-posed in Fourier-Besov-Morrey spaces $\mathcal{F}\mathcal{N}_{1,\lambda,q}^{\lambda-1}$ for $1 \leq q \leq \infty$, and globally well-posed in these spaces with small initial data. Before stating the main result of this paper, we first recall the definitions of Morrey spaces, Besov-Morrey spaces and Fourier-Besov-Morrey spaces and present some properties about these spaces. Our results on well-posedness of solutions are stated in Section 3. In Section 4, we obtain the needed linear and nonlinear estimates and we prove the well-posedness result.

II. GENERAL NOTATION

Before stating our main result, we shall introduce the notations used throughout this paper.

We denote by C a positive constant such that whose value may change with each appearance, $x \lesssim y$ means that there exists a positive constant such that $x \leq Cy$, we write $(a, b) \in X$ for $a \in X$ and $b \in X$ and $\|\cdot\|_{E \cap F} = \|\cdot\|_E + \|\cdot\|_F$. The symbol $\mathcal{S}(\mathbb{R}^3)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing complex-valued functions on \mathbb{R}^3 .

By $\hat{\varphi}$ we denote the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^3)$ in the version

$$\hat{\varphi}(x) := \mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\xi} \varphi(\xi) d\xi, \quad x \in \mathbb{R}^3.$$

and we define its inverse Fourier transform by

$$\check{\varphi}(\xi) = \mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ix\xi} \varphi(x) dx.$$

For two complex or extended real-valued measurable functions f, g on \mathbb{R}^3 , the convolution $f * g$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^3} f(x-y)g(y)dy, \quad \text{for } x \in \mathbb{R}^3.$$

III. PRELIMINARIES AND MAIN RESULTS

Let us introduce some basic knowledge on the Littlewood-Paley theory and Fourier-Besov-Morrey spaces.

Let φ, ψ be two radial positive functions such that $\text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and $\text{supp}(\psi) \subset \{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\}$

and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \text{for all } \xi \neq 0$$

and

$$\psi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \text{ for all } \xi \in \mathbb{R}^3.$$

We denote

$$\varphi_j(\xi) = \varphi(2^{-j}\xi), \quad \psi_j(\xi) = \sum_{k \leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x).$$

We define the homogeneous dyadic blocks $\dot{\Delta}_j$ and \dot{S}_j for all $u \in \mathcal{S}'(\mathbb{R}^3)$ as follows:

$$\begin{aligned} \dot{\Delta}_j u &:= \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\mathcal{F}(u)) = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x-y) dy, \\ \dot{S}_j u &:= \sum_{k \leq j-1} \Delta_k f = \mathcal{F}^{-1}(\psi(2^{-j}\xi)\mathcal{F}(u)) = 2^{3j} \int_{\mathbb{R}^3} g(2^j y) u(x-y) dy, \end{aligned}$$

where $\dot{\Delta}_j = \dot{S}_j - \dot{S}_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi| \lesssim 2^j\}$.

Then for any $u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3)$ where $\mathcal{P}(\mathbb{R}^3)$ is the set of polynomials (See. [19]) we have the Littlewood-Paley decomposition:

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \text{ and } \dot{S}_j u = \sum_{k \leq j-1} \dot{\Delta}_k u.$$

By using the definition of $\dot{\Delta}_j$ and \dot{S}_j , one easily obtains that

$$\begin{aligned} \dot{\Delta}_j \dot{\Delta}_k u &= 0, \quad \text{if } |j-k| \geq 2 \\ \dot{\Delta}_j (\dot{S}_{k-1} f \dot{\Delta}_k u) &= 0, \quad \text{if } |j-k| \geq 5. \end{aligned}$$

Now, we define the Morrey spaces $M_p^\lambda(\mathbb{R}^3)$.

Definition 3.1. ([22]) For $1 \leq p < \infty, 0 \leq \lambda < 3$, the Morrey space $M_p^\lambda = M_p^\lambda(\mathbb{R}^3)$ is defined by $M_p^\lambda(\mathbb{R}^3) = \{f \in L_{loc}^p(\mathbb{R}^3); \|f\|_{M_p^\lambda} < \infty\}$, where

$$\|f\|_{M_p^\lambda} = \sup_{x_0 \in \mathbb{R}^3} \sup_{r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0, r))}.$$

Remark 3.2. ([10])

- 1) The space M_p^λ equipped with the norm $\|\cdot\|_{M_p^\lambda}$ is a Banach space.
- 2) If $1 \leq p_1, p_2, p_3 < \infty$, $0 \leq \lambda_1, \lambda_2, \lambda_3 < 3$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have the Hölder inequality

$$\|fg\|_{M_{p_3}^{\lambda_3}} \leq \|f\|_{M_{p_1}^{\lambda_1}} \|g\|_{M_{p_2}^{\lambda_2}}.$$

- 3) For $1 \leq p < \infty$ and $0 \leq \lambda < 3$,

$$\|\varphi * g\|_{M_p^\lambda} \leq \|\varphi\|_{L^1} \|g\|_{M_p^\lambda}, \tag{2}$$

for all $\varphi \in L^1$ and $g \in M_p^\lambda$.

Lemma 3.3. ([10]) Let $1 \leq p_2 \leq p_1 < \infty$, $0 \leq \lambda_1, \lambda_2 < 3$, $\frac{3-\lambda_1}{p_1} \leq \frac{3-\lambda_2}{p_2}$ and let γ be a multi-index. If $\text{supp } p(\widehat{f}) \subset \{|\xi| \leq A2^j\}$, then there is a constant $C > 0$ independent of f and j such that

$$\left\| (i\xi)^\gamma \widehat{f} \right\|_{M_{p_2}^{\lambda_2}} \leq C 2^{j|\gamma|+j\left(\frac{3-\lambda_2}{p_2} - \frac{3-\lambda_1}{p_1}\right)} \|\widehat{f}\|_{M_{p_1}^{\lambda_1}}. \tag{3}$$

Then, we define the function spaces $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$.

Definition 3.4. ([7]) (Homogeneous Besov-Morrey spaces) Let $s \in \mathbb{R}$, $1 \leq p < +\infty$, $1 \leq q \leq +\infty$, and $0 \leq \lambda < 3$. The space $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$ is defined by $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3) = \left\{ u \in \mathcal{Z}'(\mathbb{R}^3); \|u\|_{\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)} < \infty \right\}$, where

$$\|u\|_{\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)} = \begin{cases} \left\{ \sum_{j \in \mathbb{Z}} 2^{jq_s} \|\dot{\Delta}_j u\|_{M_p^\lambda}^q \right\}^{1/q}, & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{M_p^\lambda}, & \text{for } q = \infty, \end{cases}$$

with appropriate modifications made when $q = \infty$. The space $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}(\mathbb{R}^3) : (\partial^\beta \widehat{f})(0) = 0, \text{ for every multi-index } \beta \right\}.$$

Definition 3.5. ([7]) (Homogeneous Fourier-Besov-Morrey spaces) Let $s \in \mathbb{R}$, $0 \leq \lambda < 3$, $1 \leq p < +\infty$, and $1 \leq q \leq +\infty$. The space $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$ denotes the set of all $u \in \mathcal{Z}'(\mathbb{R}^3)$ such that

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jq_s} \|\widehat{\Delta}_j u\|_{M_p^\lambda}^q \right\}^{1/q} < +\infty, \tag{4}$$

with appropriate modifications made when $q = \infty$.

Note that the space $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^3)$ equipped with the norm (4) is a Banach space. Since $M_p^0 = L^p$, we have $\mathcal{F}\dot{\mathcal{N}}_{p,0,q}^s = \mathcal{F}\dot{B}_{p,q}^s$, $\mathcal{F}\dot{\mathcal{N}}_{1,0,q}^s = \mathcal{F}\dot{B}_{1,q}^s = \dot{B}_q^s$ and $\mathcal{F}\dot{\mathcal{N}}_{1,0,1}^{-1} = \chi^{-1}$, where \dot{B}_q^s is the Fourier-Herz space, and χ^{-1} is the Lei-Lin space.

Now, we give the definition of the mixed space-time spaces.

Definition 3.6. ([7]) Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < 3$ and $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t, x)\|_{\mathcal{L}^p(0, T; \dot{\mathcal{N}}_{p, \lambda, q}^s)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jq_s} \left\| \widehat{\Delta_j u} \right\|_{\mathcal{L}^p(0, T; \dot{M}_p^\lambda)}^q \right\}^{1/q}.$$

We denote by $\mathcal{L}^p(0, T; \dot{\mathcal{N}}_{p, \lambda, q}^s)$ the set of distributions in $S'(\mathbb{R} \times \mathbb{R}^3) / \mathcal{P}(\mathbb{R}^3)$ with finite $\|\cdot\|_{\mathcal{L}^p(0, T; \dot{\mathcal{N}}_{p, \lambda, q}^s)}$ norm.

We will use the next lemma to prove our main theorem.

Lemma 3.7. ([22]) Let X be a Banach space, \mathfrak{B} a continuous bilinear map from $X \times X$ to X , and ε a positive real number such that

$$\varepsilon < \frac{1}{4\|\mathfrak{B}\|} \text{ with } \|\mathfrak{B}\| := \sup_{\|u\|, \|v\| \leq 1} \|\mathfrak{B}(u, v)\|.$$

For any y in the ball $B(0, \varepsilon)$ (ie., with center 0 and radius ε) in X , then there exists a unique x in $B(0, 2\varepsilon)$ such that

$$x = y + \mathfrak{B}(x, x).$$

Below, we shall present our main result that establishes the local and global existence.

Theorem 3.8. Let $q \in [1, +\infty]$, $\alpha \in (0, 1)$ and $0 < \lambda < 3$.

(1) For any initial data $(u_0, \omega_0) \in \mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda-\alpha}(\mathbb{R}^3)$ satisfying $\operatorname{div} u_0 = 0$, there exists a positive T such that the system (1) has a unique mild solution such that

$$(u, \omega) \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda+\alpha}(\mathbb{R}^3)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda-\alpha}(\mathbb{R}^3)\right).$$

(2) There exists a positive constant ε such that for any initial data $(u_0, \omega_0) \in \mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda-1}(\mathbb{R}^3)$ satisfying $\operatorname{div} u_0 = 0$ and

$$\|(u_0, \omega_0)\|_{\mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda-1}(\mathbb{R}^3)} < \varepsilon,$$

the system (1) has a unique global mild solution such that

$$(u, \omega) \in \mathcal{L}^{\frac{2}{1+\alpha}}\left(0, \infty; \mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda+\alpha}(\mathbb{R}^3)\right) \cap \mathcal{L}^{\frac{2}{1-\alpha}}\left(0, \infty; \mathcal{F}\dot{\mathcal{N}}_{1, \lambda, q}^{\lambda-\alpha}(\mathbb{R}^3)\right).$$

Before proving our main result we will present the corresponding linear system of the nonlinear system (1).

$$\begin{cases} \partial_t u - \Delta u - \nabla \times \omega = 0 \\ \partial_t \omega - \Delta \omega + 2\omega - \nabla \operatorname{div} \omega - \nabla \times u = 0, \\ \operatorname{div} u = 0, \\ (u, \omega)|_{t=0} = (u_0, \omega_0). \end{cases} \quad (5)$$

The solution operator of the above problem is denoted by the notation $G(t)$, i.e., for specified initial data (u_0, ω_0) in suitable function space, $(u, \omega)^T = G(t)(u_0, \omega_0)^T$ is the unique solution of the problem (5). The operator $G(t)$ has the following expression, as shown by a simple calculation:

$$(\widehat{G(t)f})(\xi) = e^{-\mathcal{A}(\xi)t} \hat{f}(\xi) \quad \text{for } f(x) = (f_1(x), f_2(x))^T,$$

where

$$\mathcal{A}(\xi) = \begin{bmatrix} |\xi|^2 I & \mathcal{B}(\xi) \\ \mathcal{B}(\xi) & (|\xi|^2 + 2)I + C(\xi) \end{bmatrix},$$

with

$$\mathcal{B}(\xi) = i \begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} \quad \text{and} \quad C(\xi) = \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2 \end{bmatrix}.$$

On the other hand, by applying the Leray projection \mathbf{P} to both sides of the first equations of (1), one can eliminate the pressure π and one check

$$\begin{cases} \partial_t u - \Delta u + \mathbf{P}(u \cdot \nabla u) - \nabla \times \omega = 0 \\ \partial_t \omega - \Delta \omega + u \cdot \nabla \omega + 2\omega - \nabla \operatorname{div} \omega - \nabla \times u = 0 \\ \operatorname{div} u = 0 \\ (u, \omega)|_{t=0} = (u_0, \omega_0), \end{cases} \quad (6)$$

where $\mathbf{P} = T + \nabla(-\Delta)^{-1} \operatorname{div}$ is the 3×3 matrix pseudo-differential operator in \mathbb{R}^3 with the symbol $(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2})_{i,j=1}^3$. We denote

$$U(x, t) = \begin{pmatrix} u(x, t) \\ \omega(x, t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u(x, 0) \\ \omega(x, 0) \end{pmatrix} = \begin{pmatrix} u_0 \\ \omega_0 \end{pmatrix}, \quad U_i(x, t) = \begin{pmatrix} u_i(x, t) \\ \omega_i(x, t) \end{pmatrix}, \quad i = 1, 2$$

and

$$U_1 \tilde{\otimes} U_2 = \begin{pmatrix} u_1 \otimes u_2 \\ u_1 \otimes \omega_2 \end{pmatrix}, \quad \tilde{\mathbf{P}} \nabla \cdot (U_1 \tilde{\otimes} U_2) = \begin{pmatrix} \mathbf{P} \nabla \cdot (u_1 \otimes u_2) \\ \nabla \cdot (u_1 \otimes \omega_2) \end{pmatrix}.$$

To solve system (6) it suffices to find the solution U of the following integral equations:

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau) \tilde{\mathbf{P}} \nabla \cdot (U \otimes U)(\tau) d\tau. \quad (7)$$

A solution of (7) is called a mild solution of (1).

IV. PROOF OF MAIN RESULT

In this section, we will establish the local and global existence and uniqueness of solution for the problem (1). For that, we prove some estimates for the semigroup $G(\cdot)$.

4.1 Linear estimates

Firstly we give the property of semigroup $G(\cdot)$.

Lemma 4.1. [9] For $t \geq 0$ and $|\xi| \neq 0$. We have

$$\|e^{-t\mathcal{A}(\xi)}\| \leq e^{-|\xi|^2 t} \quad \text{with} \quad \|e^{-t\mathcal{A}(\xi)}\| = \sup_{\|f\| \leq 1} \|e^{-t\mathcal{A}(\xi)} f\|. \quad (8)$$

Here $\|f\| = \max_i |a_i|$ with $\|f\| = \sum_{i=1}^6 a_i v_i, v_1, v_2, \dots, v_6$ are the eigenvectors for $\mathcal{A}(\xi)$.

Next, we present the linear estimates for the semigroup $G(\cdot)$.

Lemma 4.2. Let $q \in [1, +\infty], 0 < \lambda < 3$. Then there exists a positive constant C such that

$$\|G(t)U_0\|_{\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}} \leq C \|U_0\|_{\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}} \quad (9)$$

for all $t \geq 0$ and all $U_0 \in \mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}$.

Proof. By Lemma 4.1, we have

$$\begin{aligned} \|G(t)U_0\|_{\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}} &= \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda-1)jq} \left\| \mathcal{F} [G(t)\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda-1)jq} \left\| e^{-t\mathcal{A}(\xi)} \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda-1)jq} e^{-|\xi|^2 qt} \left\| \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda-1)jq} \left\| \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \\ &\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}}. \end{aligned}$$

This completes the proof of Lemma 4.2. \square

Lemma 4.3. Let $q \in [1, +\infty], 0 < \lambda < 3, \alpha \in (0; 1)$ and $T \in (0, \infty]$. Then there exists a positive constant $C = C(\alpha)$ such that

$$\|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1\pm\alpha}}(0,T;\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda\pm\alpha}(\mathbb{R}^3))} \leq C \|U_0\|_{\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}}, \quad (10)$$

for all $t \geq 0$ and all $U_0 \in \mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}$.

Proof. From Definition 3.6, it is easy to see that

$$\begin{aligned} \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1\pm\alpha}}(0,T;\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda\pm\alpha}(\mathbb{R}^3))} &= \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\| \mathcal{F} [G(t)\dot{\Delta}_j U_0] \right\|_{\mathcal{L}^{\frac{2}{1\pm\alpha}}(0,T;\mathcal{M}_1^\lambda)}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\| e^{-t\mathcal{A}(\xi)} \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{L}^{\frac{2}{1\pm\alpha}}(0,T;\mathcal{M}_1^\lambda)}^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda\pm\alpha)jq} \left\| e^{-t2^{2j}} \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \quad \square \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda\pm\alpha)jq} 2^{-(1\pm\alpha)jq} \left\| \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} 2^{(\lambda-1)jq} \left\| \mathcal{F} [\dot{\Delta}_j U_0] \right\|_{\mathcal{M}_1^\lambda}^q \right)^{\frac{1}{q}} \\ &\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{X}}_{1,\lambda,q}^{\lambda-1}}. \end{aligned}$$

This completes the proof of Lemma 4.3.

4.2 Bilinear estimates and product laws

Lemma 4.4. *Let $T > 0$, $s \in \mathbb{R}$ and $p, q, \gamma \in [1; +\infty]$ and $0 < \lambda < 3$. Then there exists a positive constant C such that*

$$\left\| \int_0^t G(t-\tau)f(\tau)d\tau \right\|_{\mathcal{L}^\gamma(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s)} \leq C \|f\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s-\frac{2}{\gamma}})}$$

for all $f \in \mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s-\frac{2}{\gamma}})$.

Proof. By Young's inequality, we obtain

$$\begin{aligned} & \left\| \int_0^t G(t-\tau)f(\tau)d\tau \right\|_{\mathcal{L}^\gamma(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s)} \\ &= \sum_{j \in \mathbb{Z}} 2^{jq} \left\| \int_0^t e^{-(t-\tau)A(\xi)} \mathcal{F} [\dot{\Delta}_j f](\tau) d\tau \right\|_{\mathcal{L}^\gamma(0,T;M_p^\lambda)}^q \Big)^{\frac{1}{q}} \\ &\leq C \sum_{j \in \mathbb{Z}} 2^{jq} \left\| \int_0^t e^{-(t-\tau)2^{2j}} \mathcal{F} [\dot{\Delta}_j f](\tau) \right\|_{M_p^\lambda}^q d\tau \Big)^{\frac{1}{q}}_{\mathcal{L}^\gamma(0,T)} \\ &\leq C \sum_{j \in \mathbb{Z}} 2^{jq(s-\frac{2}{\gamma})} \left\| \mathcal{F} [\dot{\Delta}_j f](\tau) \right\|_{L^1(0,T;M_p^\lambda)}^q \Big)^{\frac{1}{q}} \\ &\leq C \|f\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{s-\frac{2}{\gamma}})}. \end{aligned}$$

Which finish the proof. □

In the framework of homogeneous Fourier-Besov-Morrey spaces, we now gather an essential multiplication estimates.

Lemma 4.5. *Let $p, q \in [1, +\infty]$, $0 < \lambda < 3$, $T \in (0, +\infty]$ and $\alpha \in (0, 1)$. Then there exists a positive constant C such that*

$$\begin{aligned} \|uv\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^\lambda)} &\lesssim \|u\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})} \|v\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})} \\ &+ \|v\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})} \|u\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})}. \end{aligned}$$

Proof. We introduce some notations about the standard localization operators. We set

$$u_j = \dot{\Delta}_j u, \quad \dot{S}_j u = \sum_{k \leq j-1} \dot{\Delta}_k u, \quad \tilde{\Delta}_j u = \sum_{|k-j| \leq 1} \dot{\Delta}_k u, \quad j \in \mathbb{Z}.$$

Bony's decomposition for $\dot{\Delta}_j(uv)$ reads

$$\begin{aligned} \dot{\Delta}_j(uv) &= \sum_{|k-j| \leq 4} \dot{\Delta}_j(\dot{S}_{k-1} u \dot{\Delta}_k v) + \sum_{|k-j| \leq 4} \dot{\Delta}_j(\dot{S}_{k-1} v \dot{\Delta}_k u) + \sum_{k \geq j-3} \dot{\Delta}_j(\dot{\Delta}_k u \tilde{\Delta}_j v) \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Then by the triangle inequalities, we have

$$\begin{aligned} \|uv\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda})} &\leq \left(\sum_{j \in \mathbb{Z}} 2^{j\lambda q} \|\widehat{I}_1\|_{L^1(0,T;M_1^\lambda)}^q \right)^{\frac{1}{q}} + \left(\sum_{j \in \mathbb{Z}} 2^{j\lambda q} \|\widehat{I}_2\|_{L^1(0,T;M_1^\lambda)}^q \right)^{\frac{1}{q}} \\ &+ \left(\sum_{j \in \mathbb{Z}} 2^{j\lambda q} \|\widehat{I}_3\|_{L^1(0,T;M_1^\lambda)}^q \right)^{\frac{1}{q}} \\ &:= J_1 + J_2 + J_3. \end{aligned} \quad (11)$$

The terms I_1 and I_2 are symmetrical. Using Young's inequality and Hölder's inequality we have

$$\begin{aligned} 2^{j\lambda} \|\widehat{I}_1\|_{L^1(0,T;M_1^\lambda)} &\leq 2^{j\lambda} \sum_{|k-j| \leq 4} \left\| \widehat{(\dot{S}_{k-1} u \dot{\Delta}_k v)} \right\|_{L^1(0,T;M_1^\lambda)} \\ &\leq 2^{j\lambda} \sum_{|k-j| \leq 4} \|\widehat{v}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} \sum_{l \leq k-2} \|\widehat{u}_l\|_{L^{\frac{2}{1-\alpha}}(0,T;L^1)} \\ &\leq 2^{j\lambda} \sum_{|k-j| \leq 4} \|\widehat{v}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} \sum_{l \leq k-2} 2^{\lambda l} \|\widehat{u}_l\|_{L^{\frac{2}{1-\alpha}}(0,T;M_1^\lambda)} \\ &\leq 2^{j\lambda} \sum_{|k-j| \leq 4} \|\widehat{v}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} \left(\sum_{l \leq k-2} 2^{(\lambda-\alpha)l q} \|\widehat{u}_l\|_{L^{\frac{2}{1-\alpha}}(0,T;M_1^\lambda)} \right)^{\frac{1}{q}} \left(\sum_{l \leq k-2} 2^{l\alpha q'} \right)^{\frac{1}{q'}} \\ &\leq 2^{j\lambda} \sum_{|k-j| \leq 4} 2^{\alpha k} \|\widehat{v}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} \|u\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{-\alpha+\lambda})} \\ &\leq \sum_{|k-j| \leq 4} 2^{(\alpha+\lambda)k} 2^{(j-k)\lambda} \|\widehat{v}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} \|u\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})}. \end{aligned}$$

Taking ℓ^q -norm we get

$$J_1 \leq \|v\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})} \|u\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})},$$

and in a similar way we obtain

$$J_2 \leq \|u\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})} \|v\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})}.$$

To estimate I_3 , let

$$I_3^k = \dot{\Delta}_j \left(\sum_{|k'-k|} \dot{\Delta}_{k'} v \dot{\Delta}_k u \right) = \sum_{k'=-1}^1 \dot{\Delta}_k u \dot{\Delta}_{k+k'} v.$$

First we use Young's inequality (2) in Morrey spaces, and Lemma 3.3 with $|\gamma|=0$, to obtain

$$\begin{aligned} 2^{j\lambda} \|\widehat{I}_3\|_{L^1(0,T;M_1^\lambda)} &\leq 2^{j\lambda} \sum_{k \geq j-3} \|\widehat{I}_3^k\|_{L^1(0,T;M_1^\lambda)} \\ &\leq 2^{j\lambda} \sum_{k \geq j-3} \sum_{|k'-k| \leq 1} \|\widehat{u}_k\|_{L^{\frac{2}{1-\alpha}}(0,T;M_1^\lambda)} \|\widehat{v}_{k'}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^1)} \\ &\leq 2^{j\lambda} \sum_{k \geq j-3} \sum_{|k'-k| \leq 1} \|\widehat{u}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} 2^{\lambda k'} \|\widehat{v}_{k'}\|_{L^{\frac{2}{1-\alpha}}(0,T;M_1^\lambda)} \\ &\leq 2^{j\lambda} \sum_{k \geq j-3} \sum_{|k'-k| \leq 1} 2^{k'(\lambda-\alpha)q} \|\widehat{v}_{k'}\|_{L^{\frac{2}{1-\alpha}}(0,T;M_1^\lambda)} \left(\sum_{k \geq j-3} 2^{\alpha k} \|\widehat{u}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)} \right)^{\frac{1}{q}} \\ &\leq C \|v\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})} \sum_{k \geq j-3} 2^{(\lambda+\alpha)k} 2^{\lambda(j-k)} \|\widehat{u}_k\|_{L^{\frac{2}{1+\alpha}}(0,T;M_1^\lambda)}, \end{aligned} \quad (12)$$

taking ℓ^q -norm on both sides in the above estimate, we get

$$J_3 \leq \|v\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})} \|u\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})}.$$

Which gives the result. □

Below, we shall prove our result.

4.3 Proof of Theorem 3.8

1) Let $T > 0$ and $\alpha \in (0; 1)$ we define the space X_T^α as

$$X_T^\alpha = \left\{ U : U \in \mathcal{L}^{\frac{2}{1+\alpha}} \left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha}(\mathbb{R}^3) \right) \cap \mathcal{L}^{\frac{2}{1-\alpha}} \left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha}(\mathbb{R}^3) \right) \right\}$$

and equipped with the following norm:

$$\|U\|_{X_T^\alpha} = \|U\|_{\mathcal{L}^{\frac{2}{1+\alpha}} \left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha}(\mathbb{R}^3) \right)} + \|U\|_{\mathcal{L}^{\frac{2}{1-\alpha}} \left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha}(\mathbb{R}^3) \right)}.$$

For all $U \in X_T^\alpha$ we define $\phi(U)$ as follows

$$\phi(U) = G(t)U_0 - \int_0^t G(t-\tau)\tilde{\mathbf{P}}\nabla \cdot (U \otimes U)(\tau) d\tau. \tag{13}$$

Our goal is to show that U is a fixed point of ϕ .

Considering

$$B(U_1, U_2) = \int_0^t G(t-\tau)\tilde{\mathbf{P}}\nabla \cdot (U_1 \otimes U_2)(\tau) d\tau. \tag{14}$$

Then, by Lemma 4.4, Lemma 4.5 and the embedding $\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^\lambda \hookrightarrow \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}$, we have

$$\begin{aligned} \|B(U_1, U_2)\|_{X_T^\alpha} &= \left\| \int_0^t G(t-\tau)\tilde{\mathbf{P}}\nabla \cdot (U_1 \otimes U_2)(\tau) d\tau \right\|_{\mathcal{L}^{\frac{2}{1+\alpha}} \left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha}(\mathbb{R}^3) \right)} \\ &\quad + \left\| \int_0^t G(t-\tau)\tilde{\mathbf{P}}\nabla \cdot (U_1 \otimes U_2)(\tau) d\tau \right\|_{\mathcal{L}^{\frac{2}{1-\alpha}} \left(0, T; \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha}(\mathbb{R}^3) \right)} \\ &\lesssim \|\tilde{\mathbf{P}}\nabla \cdot (U_1 \otimes U_2)\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1})} \\ &\lesssim \|\tilde{\mathbf{P}}\nabla \cdot (U_1 \otimes U_2)\|_{\mathcal{L}^1(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^\lambda)} \\ &\lesssim \|U_1\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})} \|U_2\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})} + \|U_2\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha})} \|U_1\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha})} \\ &\leq C_1 \|U_1\|_{X_T^\alpha} \|U_2\|_{X_T^\alpha}. \end{aligned} \tag{15}$$

Then, by (13) and (15), one concludes

$$\|\phi(U)\|_{X_T^\alpha} \leq \|G(t)U_0\|_{X_T^\alpha} + C_1 \|U_1\|_{X_T^\alpha} \|U_2\|_{X_T^\alpha}.$$

By Lemma 4.3, we get

$$\|G(t)U_0\|_{X_T^\alpha} \leq C_2 \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}}. \tag{16}$$

Now, consider the norm $\|G(t)U_0\|_{X_T^\alpha}$. Using the given expression for $G(t)$ and the definition of X_T^α , we can write:

$$\|G(t)U_0\|_{X_T^\alpha} = \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha}(\mathbb{R}^3))} + \|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1-\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-\alpha}(\mathbb{R}^3))}$$

Now, let's analyze each term separately. First, note that $G(t)$ is a linear operator, so we can factor out U_0 from the norm:

$$\|G(t)U_0\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T;\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha}(\mathbb{R}^3))} \leq \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda+\alpha}(\mathbb{R}^3)} \|G(t)\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T)}.$$

Similarly for the second term. Now, for sufficiently small T , $e^{-\mathcal{A}(\xi)t}$ tends to I (the identity operator) as t approaches 0. Therefore, for small T , $\|G(t)\|_{\mathcal{L}^{\frac{2}{1+\alpha}}(0,T)}$ tends to 0.

Then $\|G(t)U_0\|_{X_T^\alpha} \rightarrow 0$ as $T \rightarrow 0$, hence $\alpha \neq 1$, and there exists $T > 0$ such that $\|G(t)U_0\|_{X_T^\alpha} < \frac{1}{4C_1}$. Using Lemma 3.7, system (1) admits a unique mild solution $U \in X_T^\alpha$ with $\|U\|_{X_T^\alpha} < \frac{1}{2C_1}$.

For 2) we replace X_T^α by X_∞^α and we get

$$\|B(U_1, U_2)\|_{X_\infty^\alpha} \leq C_1 \|U_1\|_{X_\infty^\alpha} \|U_2\|_{X_\infty^\alpha}. \quad (17)$$

Then, by (16) and (17), one obtains

$$\|\Phi(U)\|_{X_\infty^\alpha} \leq C_2 \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}} + C_1 \|U_1\|_{X_\infty^\alpha} \|U_2\|_{X_\infty^\alpha}.$$

Then, by applying Lemma 3.7, with $\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}} < \frac{1}{4C_1C_2}$. Then, system (1) admits a unique global mild solution $U \in X_\infty^\alpha$ with $\|U\|_{X_\infty^\alpha} < \frac{1}{2C_1}$. This completes the proof of Theorem 3.8 (2).

REFERENCES

1. A. Azanzal, A. Abbassi, C. Allalou, On the Cauchy problem for the fractional drift-diffusion system in critical Fourier-Besov-Morrey spaces. *International Journal On Optimization and Applications*, 1(2021), p-28.
2. A. Azanzal, A. Abbassi, C. Allalou, Existence of Solutions for the Debye-Hückel System with Low Regularity Initial Data in Critical Fourier-Besov-Morrey Spaces. *Nonlinear Dynamics and Systems Theory*, 21, 367-380 (2021).
3. A. Azanzal, A. Abbassi, C. Allalou, S. Melliani, Well-posedness and blow-up of solutions for the 2D dissipative quasi-geostrophic equation in critical Fourier-Besov-Morrey spaces. *J Elliptic Parabol Equ.* <https://doi.org/10.1007/s41808-021-00140-x> (2021).
4. A. Azanzal, C. Allalou, A. Abbassi, Well-posedness and analyticity for generalized Navier-Stokes equations in critical Fourier-Besov-Morrey spaces. *J. Nonlinear Funct. Anal.* 2021(2021), Article ID 24.
5. Q. Chen, C. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, *J. Differential Equations* 252 (3) (2012) 2698-2724.
6. B. Dong, J. Li, J. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, *J. Differential Equations* 262 (2017) 3488-3523
7. A. El Baraka, M. Toumlilin, Uniform well-Posedness and stability for fractional Navier-Stokes Equations with Coriolis force in critical Fourier-Besov-Morrey Spaces, *Open J. Math. Anal.* 2019, 3(1), 70-89.
8. A.C. Eringen, Theory of micropolar fluids, *J. Math. Mech.* 16 (1966) 1-18.
9. L.C. Ferreira, E.J. Villamizar-Roa, Micropolar fluid system in a space of distributions and large time behavior, *J. Math. Anal. Appl.* 332 (2) (2007) 1425-1445.

10. L. C. Ferreira, L. S. Lima; Self-similar solutions for active scalar equations in Fourier-BesovMorrey spaces, *Monatsh. Math.*, 175.4 (2014), 491-509.
11. GP. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fluid equations. *International Journal of Engineering Science* 1977; 15:105-108.
12. H. Inoue, Matsuura M. Ôtani, Strong solutions of magneto-micropolar fluid equation. *Discrete and Continuous Dynamical Systems* 2003; 439-448; *Dynamical systems and differential equations* (Wilmington, NC, 2002).
13. Z. Lei, F. Lin, Global mild solutions of Navier-Stokes equations, *Comm. Pure Appl. Math.*, 64.9 (2011), 1297-1304.
14. P.G. Lemarié-Rieusset, The role of Morrey spaces in the study of Navier-Stokes and Euler equations. *Eurasian Math. J.* 3, 62-93 (2012).
15. G. Lukaszewicz On nonstationary flows of asymmetric fluids. *Rendiconti Accademia Nazionale delle Scienze della dei XL Serie V. Memorie di Matematica* 1988; 12:83-97.
16. F. Ouidirne, H. Srhiri, C. Allalou, and M. Oukessou, Global existence for the 3-D generalized micropolar fluid system in critical Fourier-Besov spaces with variable exponents. *Nonlinear J. Dynamics and Systems Theory.* 23(3):338-347
17. H. Srhiri, F. Ouidirne, C. Allalou, and K. Hilal, Well-posedness and stability of solutions for the 3-D generalized micropolar system in Fourier-BesovMorrey spaces. *Journal of Elliptic and Parabolic Equations*, 1-31, (2023).
18. J. Sun, S. Cui, Sharp well-posedness and ill-posedness in Fourier-Besov spaces for the viscous primitive equations of geophysics, *arXiv:1510.0713v1*, (2015).
19. H. Triebel, *Theory of Function Spaces*, in: *Monogr. Math.*, vol. 78, Birkhäuser Verlag, Basel, 1983.
20. E.J. Villamizar-Roa, M.A. Rodríguez-Bellido, Global existence and exponential stability for the micropolar fluid system, *Z. Angew. Math. Phys.* 59 (5) (2008) 790-809.
21. W. Zhu, J. Zhao, Existence and regularizing rate estimates of solutions to the 3-D generalized micropolar system in Fourier- Besov spaces. *Math. Methods Appl. Sci.* 41 (4) (2018) 1703- 1722.
22. W. Zhu, Sharp well-posedness and ill-posedness for the 3-D micropolar fluid system in FourierBesov spaces. *Nonlinear Analysis: Real World Applications* 46 (2019) 335-351.



Scan to know paper details and
author's profile

An Investigation into the Order of Integral Powers of Consecutive Elements of Set of Even and Odd Numbers

*Ladan, Umaru Ibrahim, Emmanuel, J. D. Garba & Tanko Ishaya
& Ibrahim Abdullahi Muhammad (Ibzar)*

University of Jos, Ahmadu Bello University Zaria

ABSTRACT

This article analysed the order of difference of integral perfect powers of the set of even and odd numbers. The analysis was proof by the use of combinatorial terminologies, established property of the difference operator and the principle of mathematical induction. The results proved conclusively that “if any number of consecutive odd or even integers are raised to a positive power k , then the k th difference is equal to $2^k k!$ ”

Keywords: finite difference, mathematical induction, integral order, positive powers, reproductive property.

Classification: LCC Code: QA214

Language: English



Great Britain
Journals Press

LJP Copyright ID: 925613

Print ISSN: 2631-8490

Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0



An Investigation into the Order of Integral Powers of Consecutive Elements of Set of Even and Odd Numbers

Ladan, Umaru Ibrahim^α, Emmanuel, J. D. Garba^σ, Tanko Ishaya^ρ & Ibrahim Abdullahi Muhammad (Ibzar)^ϐ

ABSTRACT

This article analysed the order of difference of integral perfect powers of the set of even and odd numbers. The analysis was proof by the use of combinatorial terminologies, established property of the difference operator and the principle of mathematical induction. The results proved conclusively that “if any number of consecutive odd or even integers are raised to a positive power k , then the k th difference is equal to $2^k k!$ ”

Keywords: finite difference, mathematical induction, integral order, positive powers, reproductive property.

Author α: Department of Computer Science, Faculty of Natural Sciences, University of Jos.

σ: Department of Mathematics, Faculty of Natural Sciences, University of Jos.

ρ: Department of Computer science, Faculty of Natural Sciences, University of Jos

ϐ: Department of Political science, Faculty of Social Sciences, Ahmadu Bello University Zaria.

I. INTRODUCTION

A perfect power is a number which has a rational root. Chase^[1]. An integral perfect power is the irrational root that is an integer. A finite difference is a mathematical expression of the form, $f(x + b) - f(x + a)$ and a forward difference is of the form $\Delta_h [f](x) = f(x + h) - f(x)$ ^[2].

In this article, an examination on the set of even numbers: $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$ and the set of odd numbers: $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ reveals that the first difference of the square of consecutive elements of both the set of even and odd numbers is even. The second difference of the two distinct sets is $\{8, 8, 8, 8, 8, 8, 8, 8, 8\} = \{8\}$, a singleton set.

Clearly, the third difference is $\{0\}$. The emerging pattern motivates the investigation of whether or not this pattern persists for all perfect squares of elements of even and odd sets, resulting from integers. In their contribution in Exponential Diophantine Equations, Shorey and Tijdeman^[3] investigated perfect powers at integral values of a polynomial with rational integer coefficients and obtain in particular the following.

Let $f(x)$ be a polynomial with rational coefficients and with at least two simple rational zeros.

Suppose $b \neq 0$, $m \geq 0$, x and y with $|y| > 1$ are rational integers. Then the equation $f(x) = by^m$ implies that m is bounded by a computable number depending only on b and f . Also, in their contribution in the difference of perfect powers of integers, Ladan, Ukwu and Apine^[4] investigated order of integral perfect powers and proved that “if any number of consecutive integers are raised to a positive integral power k , then the k^{th} difference is equal to $k!$ ”

More generally, the question at the heart of the matter is the following. What is the computational disposition of the orders of difference of integral perfect powers of successive elements of set of even and odd numbers? The Review of Literature shows that no such investigation has been undertaken. Thus, this article adds to the existing body of knowledge, by providing answers to the above question.

II. METHODS

2.1 Preliminary Definitions

In what follows, the difference of finite order will be defined.

2.1.1 Difference of Order One (1)

Given a sequence $\{U_j\}_{j=0}^{\infty}$, define the difference of order one at j with respect to the sequence by:

$$\Delta(U_j) = U_{j+1} - U_j, \text{ for every integral } j.$$

2.1.2 Higher Order Difference

Higher difference can be defined recursively by:

$$\Delta^k(U_j) = \Delta \Delta^{k-1}(U_j) = \Delta^{k-1} \Delta(U_j) = \Delta^{k-1}[U_{j+1} - U_j] \text{ for } k \geq 2.$$

III. RESULTS AND DISCUSSION

3.1 Preliminary Theorem

Suppose that $U_j = j$ and $\hat{U}_j = \hat{j}$, for integral j .

Let $\alpha_j U_j = \alpha_j j$ be the set of even numbers and $\alpha_j \hat{U}_j + \beta_j = \alpha_j j + \beta_j$ be the set of odd numbers.

Where $\alpha_j=2$ fixed, $j \geq 0$, $\beta = 1$, fixed.

Then,

- (i)₁ $\Delta(\alpha_j U_j)^2 = \alpha_j^2(\text{odd})$ for even case
- (i)₂ $\Delta(\alpha_j \hat{U}_j + \beta)^2 = \alpha_j^2(\text{even})$ for odd case
- (ii) $\Delta^2(\alpha_j U_j)^2 = \Delta^2(\alpha_j \hat{U}_j + \beta)^2 = 8 = 2^2(2!)$
- (iii) $\Delta^k(\alpha_j U_j)^2 = \Delta^k(\alpha_j \hat{U}_j + \beta)^2 = 0, k \in \{3, 4, \dots\}$

Proof:

Case 1. (even)

$$\text{Let } \langle \alpha_j U_j \rangle_{j=0}^9 = \langle \alpha_j j \rangle_{j=0}^9 = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

Then, the table below display the values of $\Delta^k(\alpha_j U)^P$ for selected values of k and P in the set $\{0, 1, 2, 3\}$ and $\{1, 2\}$ respectively when $\Delta^0(\alpha_j U_j)^P = (\alpha_j U_j)^P$.

Table I: Difference Order Table for Selected Consecutive Positive Integers.

| α_j | $(\alpha_j)^2$ | $\Delta(\alpha_j)^2$ | $\Delta^2(\alpha_j)^2$ | $\Delta^3(\alpha_j)^2$ |
|------------|----------------|----------------------|------------------------|------------------------|
| 0 | 0 | 4 | 8 | 0 |
| 2 | 4 | 12 | 8 | 0 |
| 4 | 16 | 20 | 8 | 0 |
| 6 | 36 | 28 | 8 | 0 |
| 8 | 64 | 36 | 8 | 0 |
| 10 | 100 | 44 | 8 | 0 |
| 12 | 144 | 52 | 8 | 0 |
| 14 | 196 | 60 | 8 | 0 |
| 16 | 256 | 68 | 8 | |
| 18 | 324 | | | |

It is clear that $\Delta^k(\alpha_j)^2 = 0$, for all $k \geq 3$, $\alpha_j = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$. Obviously the theorem is valid for $j \geq 0$ as observed from columns 4 and 5.

Case II. (odd)

Let $\langle \alpha_j + \beta \rangle_{j=0}^9 = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

Then the table below displayed the values of $\Delta^k(\alpha_j + \beta_j)^P$ for selected values of k and p in the set $\{0, 1, 2, 3\}$ and $\{1, 2\}$ respectively when $\Delta^0(\alpha_j + \beta_j)^P = (\alpha_j + \beta_j)^P$.

Table II: Difference Order table for selected Consecutive Positive Integers

| $\alpha_j + \beta$ | $(\alpha_j + \beta)^2$ | $\Delta(\alpha_j + \beta)^2$ | $\Delta^2(\alpha_j + \beta)^2$ | $\Delta^3(\alpha_j + \beta)^2$ |
|--------------------|------------------------|------------------------------|--------------------------------|--------------------------------|
| 1 | 1 | 8 | 8 | 0 |
| 3 | 9 | 16 | 8 | 0 |
| 5 | 25 | 24 | 8 | 0 |
| 7 | 49 | 32 | 8 | 0 |
| 9 | 81 | 40 | 8 | 0 |
| 11 | 121 | 48 | 8 | 0 |
| 13 | 169 | 56 | 8 | 0 |
| 15 | 225 | 64 | 8 | |
| 17 | 289 | 72 | | |
| 19 | 361 | | | |

Similarly, it is clear that $\Delta^k(\alpha_j + \beta_j) = 0$, for all $k \geq 3$, $\alpha_j + \beta = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$. The theorem is valid as observed from column 4 and 5.

The proof of (i), (ii) and (iii) are direct.

$$\begin{aligned}
 (i_1) \quad U_j = j &\Rightarrow \alpha U_j = \alpha_j \Rightarrow (\alpha U_j)^2 = \alpha^2 j^2 \\
 &\Rightarrow (\alpha^2 j^2) = \alpha^2 \Delta j^2 = \alpha^2 [(j+1)^2 - j^2] = \alpha^2 [j^2 + 2j + 1 - j^2] \\
 &= \alpha^2 (2j + 1) = \alpha^2 (\text{odd}) = \text{even}.
 \end{aligned}$$

$$\begin{aligned}
 (i_2) \quad \widehat{U}_j = \widehat{j} &\Rightarrow \alpha \widehat{U}_j + \beta = \alpha \widehat{j} + \beta \\
 (\widehat{U}_j + \beta)^2 &= (U_j + \beta)^2 = (\alpha_j)^2 + 2\alpha \beta_j + \beta^2 \\
 \Delta(\alpha \widehat{U}_j + \beta_j)^2 &= \Delta(\alpha^2 j^2 + 2\alpha \beta_j + \beta^2) \\
 &= \alpha^2 \Delta j^2 + 2\alpha \beta \Delta_j + \beta^2 \\
 &= \alpha^2 [(j+1)^2 - j^2] + 2\alpha \beta [(j+1) - j] + 0 \\
 &= \alpha^2 (j^2 + 2j + 1 - j^2) + 2\alpha \beta \quad (1) \\
 &= \alpha^2 (2j + 1) + 2\alpha \beta \\
 &= 2\alpha^2 j + \alpha^2 + 2\alpha \beta \\
 &= \alpha^2 (2j + 1) + 2\alpha \beta \\
 &\Rightarrow \Delta(\alpha \widehat{U}_j + \beta_j)^2 = \alpha^2 (2j + 1) + 2\alpha \beta = \text{even}
 \end{aligned}$$

For every $j \geq 0$, $\alpha = 2$, $\beta = 1$.

$$\begin{aligned}
 (ii) \quad \Delta^2(\alpha U_j)^2 &= \Delta(\Delta(\alpha j))^2 = \Delta[\alpha^2 (2j + 1)] \\
 &= \Delta(2\alpha^2 j + \alpha^2) = 2\alpha^2 \Delta j + \Delta\alpha^2 \\
 &= 2\alpha^2 [(j+1) - j] + 0 \\
 &= 2\alpha^2 (1) = 2\alpha^2 \\
 &\Rightarrow \Delta^2(\alpha U_j)^2 = 2\alpha^2
 \end{aligned}$$

Similarly:

$$\begin{aligned}
 \Delta^2(\alpha \widehat{U}_j + \beta)^2 &= \Delta[\Delta(\alpha j + \beta)^2] \\
 &= \Delta[\alpha^2 (2j + 1) + 2\alpha \beta] \\
 &= \Delta[2\alpha^2 j + \alpha^2 + 2\alpha \beta]
 \end{aligned}$$

$$\begin{aligned}
&= 2 \alpha^2 \Delta j + \Delta \alpha^2 + 2 \Delta \alpha \beta \\
&= 2 \alpha^2 [(j + 1) - j] + 0 + 0 \\
&= 2 \alpha^2 (1) = 2 \alpha^2
\end{aligned}$$

$$\Rightarrow \Delta^2(\alpha \hat{U}_j + \beta) = 2 \alpha^2$$

$$\Rightarrow \Delta^2(\alpha U_j)^2 = \Delta^2(\alpha \hat{U}_j + \beta) = 2 \alpha^2. \text{ proving (ii)}$$

The principle of mathematical induction is needed in the proof of (iii).

$$\text{For } k = 3, \Delta^3(\alpha_j U_j)^2 = \Delta^3(\alpha j)^2 = \Delta[\Delta^2(\alpha j)^2] = \Delta(2 \alpha^2) = 0$$

$$\text{Similarly, } \Delta^3(\alpha \hat{U}_j + \beta)^2 = \Delta^3(\alpha j + \beta)^2 = \Delta[\Delta^2(\alpha j + \beta)^2] = \Delta(2 \alpha^2) = 0$$

$$\text{Consequently, } \Delta^k(\alpha \hat{U}_j)^2 = \Delta^{k-3}[\Delta^3(\alpha j)^2] = \Delta^{k-3}(0) = 0$$

$$\text{and } \Delta^k(\alpha \hat{U}_j + \beta) = \Delta^{k-3}[\Delta^3(\alpha j + \beta)^2] = \Delta^{k-3}(0) = 0$$

proving $\Delta^k(\alpha U_j)^2 = \Delta^k(\alpha \hat{U}_j + \beta)^2 = 0$, for all positive integer $k \geq 3$. This established the proof of (iii).

Thus, we have seen clearly that:

$$\text{Theorem 3.1 } \Rightarrow \Delta(\alpha U_j)^2 \text{ and } \Delta(\alpha \hat{U}_j + \beta)^2 \text{ are even (i)}$$

$$\Delta^2(\alpha U_j)^2 \text{ and } \Delta^2(\alpha \hat{U}_j + \beta)^2 = \alpha^2 (2!), \alpha = 2 \text{ (ii)}$$

$$\Delta^3(\alpha U_j)^2 \text{ and } \Delta^3(\alpha \hat{U}_j + \beta)^2 = 0 \text{ } \forall k \geq 3 \text{ (iii)}$$

In the sequel, we examine the computational disposition of $\Delta^k(\alpha U_j)^p$ and $\Delta^k(\alpha \hat{U}_j + \beta)^p$

for every integral j and positive integral k and p . The results are summarized as in the following theorem.

3.2 Main Theorem

Let $\alpha U_j = \alpha j$ and $\alpha \hat{U}_j + \beta = \alpha j + \beta$, where j is any integer. Then for arbitrary positive integer k and p , $\Delta^k(\alpha U_j)^p$ and $\Delta^k(\alpha \hat{U}_j + \beta)^p$ is given by:

$$\Delta^k(\alpha U_j)^p = \Delta^k(\alpha \hat{U}_j + \beta)^p \begin{cases} 0, & \text{if } k > p & \text{(a)} \\ \sum_{i=1}^p \binom{p}{i} j^i, & \text{for } k = 1 & \text{(b)} \\ \alpha^k k!, & \text{if } k = p & \text{(c)} \\ \text{even,} & \text{for } p \geq 1 & \text{(d)} \\ \text{even,} & \text{for } 2 \leq k < p & \text{(e)} \end{cases}$$

3.2.1 Proof of (a)

$$\begin{aligned} \Delta^2(\alpha U_j) &= \Delta(\Delta \alpha j) = \Delta[\alpha \Delta(j)] = \Delta[(j + 1) - j] \\ &= \Delta[\alpha (1)] = \Delta(\alpha) = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} \Delta^2(\alpha U_j + \beta) &= \Delta[\Delta(\alpha j + \beta)] = \Delta[\alpha \Delta j + \Delta \beta] \\ &= \Delta[\alpha ((j + 1) - j)] = \Delta[\alpha (1)] \\ &= \Delta(\alpha) = 0 \end{aligned}$$

$$\Rightarrow \Delta^2(\alpha U_j) = \Delta^2(\alpha U_j + \beta) = 0.$$

So (a) is valid for $k = 2$ and $p = 1$, which are the least values for the respective exponents. Assume that (a) is valid for all pairs of integers \hat{k}, \hat{p} for which $\hat{k} + \hat{p} \leq k + p$ for some positive integers \hat{k} and \hat{p} such that $p \geq 1, k \geq 2, p < k$.

$$\text{Then } \Delta^k(\alpha U_j)^p = \Delta[\Delta^k(\alpha U_j + \beta)^p] = \Delta(0) = 0$$

$$\text{and } \Delta^{k+1}(\alpha U_j + \beta) = \Delta[\Delta^k(\alpha U_j + \beta)] = \Delta(0) = 0$$

(by induction hypothesis hypothesis) = 0. Therefore, the validity of (a) is established.

3.2.2 Proof of (b)

$$\begin{aligned} (j + 1)^p &= \sum_{i=1}^p \binom{p}{i} j^i = \sum_{i=1}^{p-1} \binom{p}{i} j^i + \binom{p}{p} j^p \\ \Rightarrow (j + 1)^p - j^p &= \sum_{i=1}^{p-1} \binom{p}{i} j^i \text{ proving (b)} \end{aligned}$$

$$\Rightarrow \Delta^1(\alpha U_j)^p = \alpha^p (\Delta U_j^p) = \alpha^p \sum_{i=1}^{p-1} \binom{p}{i} j^i$$

$$\text{and } \Delta^1(\alpha U_j + \beta)^p = \sum_{i=1}^{p-1} \binom{p}{i} (\alpha j)^{p-i} = \alpha^p \sum_{i=1}^{p-1} \binom{p}{i} j^i$$

Observe that, since $\beta = 1$, implies that:

$$\alpha \hat{U}_j + \beta = \alpha j + 1$$

$$\therefore (\alpha j + \beta)^p = \sum_{i=1}^{p-1} \binom{p}{i} (\alpha j)^{p-i} \beta^i = \sum_{i=1}^{p-1} \binom{p}{i} (\alpha j)^{p-i}$$

For simplicity of complexity of the odd form, we can logically express it as a single form with respect to even form, since they have same characteristic structures.

Claim that proof of even case is necessary and sufficient for the proof of the odd case.

Considering the even form for the remaining part of the proof \because proof even \Leftrightarrow proof of odd.

We have that:

$$\begin{aligned} \Delta(\alpha U_j)^p &= \Delta(\alpha^p U_j^p) = \alpha^p (\Delta U_j^p) = \alpha^p (\Delta j^p) \\ &= \alpha^p [(j + 1)^p - j^p] = \alpha^p \sum_{i=1}^{p-1} \binom{p}{i} j^i \end{aligned}$$

3.2.3 Proof of (c)

We examine $\Delta^k(\alpha U_j)^p : k=1$

$$\Rightarrow (\alpha U_j) = \alpha (\Delta j) = \alpha (j + 1 - j) = \alpha (1) = \alpha = 2!$$

$$\begin{aligned} k = 2, \Delta^2(\alpha U_j)^2 &= \alpha^2 \Delta^2 U_j^2 = \Delta \alpha^2 (\Delta j^2) = \alpha^2 \Delta[(j + 1)^2 - j^2] \\ &= \alpha^2 \Delta(j^2 + 2j + 1 - j^2) = \alpha^2 \Delta(2j + 1) \\ &= \alpha^2 \Delta(2j + 1) \\ &= 2 \alpha^2 \Delta j + \Delta \alpha^2 \\ &= 2 \alpha^2 (j + 1 - j) = 0 \\ &= 2 \alpha^2 = \alpha^2 2! = 2^2 \cdot 2! \end{aligned}$$

$$\Rightarrow \Delta^2(\alpha U_j)^2 = \Delta^2(\alpha \hat{U}_j + \beta) = \alpha^2 2! = 2^2 \cdot 2! = 8.$$

by (ii) of theorem 3.1 \Rightarrow the theorem is valid for $k \in \{1, 2\}$. Assume the validity of the theorem for $k \in \{3, \dots, q\}$ for some integer $q \geq 4$. Then $\Delta^q(\alpha^q U_j^q) = \alpha^q q! \Rightarrow \Delta^q(\alpha^q U_j^q) = \alpha^q (\alpha^q U_j^q) = \alpha^q q!$

by induction hypothesis. Finally, we need to prove that:

$$\Delta^{q+1}(\alpha^{q+1} U_j^{q+1}) = \alpha^{q+1} (q + 1)!$$

Claim Δ^r is reproductive. Reproductive is understood to mean the following:

$$\Delta^r \sum_{j=1}^n \hat{\alpha}_j g_j(x) = \sum_{j=1}^n \hat{\alpha}_j \Delta^r(g_j(x))$$

Where α_j are arbitrary constants.

3.2.4 Proof of Claim

Consider $\Delta^r(\hat{\alpha}_1 U_{j_1}^{k_1} + \hat{\alpha}_2 U_{j_2}^{k_2}), r = \{1, 2, \dots\}$.

$$\begin{aligned} \text{for } r = 1, \Delta(\hat{\alpha}_1 U_{j_1}^{k_1} + \hat{\alpha}_2 U_{j_2}^{k_2}) &= \Delta(\hat{\alpha}_1 U_{j_1}^{k_1} + \hat{\alpha}_2 U_{j_2}^{k_2}) \\ &= \hat{\alpha}_1 (j + k)^{k_1} + \hat{\alpha}_2 (j + k)^{k_2} - \hat{\alpha}_1 U_{j_1}^{k_1} - \hat{\alpha}_2 U_{j_2}^{k_2} \\ &= \hat{\alpha}_1 [(j_1 + 1)^{k_1} - j_1^{k_1}] + \hat{\alpha}_2 [(j_2 + 1)^{k_2} - j_2^{k_2}] \\ &= \hat{\alpha}_1 \Delta(j_1^{k_1}) + \hat{\alpha}_2 \Delta(j_2^{k_2}) \\ &= \hat{\alpha}_1 \Delta(U_{j_1}^{k_1}) + \hat{\alpha}_2 \Delta(U_{j_2}^{k_2}) \end{aligned}$$

$\Rightarrow \Delta$ is a reproductive \Rightarrow the claim is valid for $2 \leq r \leq t$, for some integer $t \geq 3$.

$$\text{Then } \Delta^t(\hat{\alpha}_1 U_{j_1}^{k_1} + \hat{\alpha}_2 U_{j_2}^{k_2}) = \hat{\alpha}_1 \Delta^t(U_{j_1}^{k_1}) + \hat{\alpha}_2 \Delta^t(U_{j_2}^{k_2})$$

Finally,

$$\begin{aligned} \Delta^{t+1}(\hat{\alpha}_1 U_{j_1}^{k_1} + \hat{\alpha}_2 U_{j_2}^{k_2}) &= \Delta^t[\Delta(\hat{\alpha}_1 U_{j_1}^{k_1} + \hat{\alpha}_2 U_{j_2}^{k_2})] \\ &= \Delta^t(\hat{\alpha}_1 \Delta U_{j_1}^{k_1} + \hat{\alpha}_2 \Delta U_{j_2}^{k_2}) = \hat{\alpha}_1 \Delta^t(\Delta U_{j_1}^{k_1}) + \hat{\alpha}_2 \Delta^t(\Delta U_{j_2}^{k_2}) \end{aligned}$$

(by induction hypothesis)

$$= \hat{\alpha}_1 \Delta^{t+1}(U_{j_1}^{k_1}) + \hat{\alpha}_2 \Delta^{t+1}(U_{j_2}^{k_2})$$

So the theorem is valid for $r = t+1$ and hence valid for all positive integer r .

$$\begin{aligned} \text{Now, } \Delta^{q+1}(\alpha^{q+1} U_j^{q+1}) &= \Delta^q[\alpha^{q+1} \Delta(U_j^{q+1})] \\ &= \Delta^q[\alpha^{q+1} ((j + 1)^{q+1} - j^{q+1})] \end{aligned}$$

$$\begin{aligned}
 &= \Delta^q \left[\alpha^{q+1} \sum_{i=1}^q \binom{q+1}{i} j^i \right] \\
 &= \Delta^q \left[\alpha^{q+1} \sum_{i=1}^q \binom{q+1}{i} U_j^i \right] \\
 &= \alpha^{q+1} \sum_{i=1}^q \binom{q+1}{i} \Delta^q (U_j^i) \quad (\text{by reproductive property}) \\
 &= \alpha^{q+1} \binom{q+1}{i} \Delta^q (U_j^i) \quad (\text{by (a) of theorem 3.2}) \\
 &= \alpha^{q+1} \binom{q+1}{i} q! \quad (\text{by the induction hypothesis}) \\
 &= \alpha^{q+1} \binom{q+1}{i} !
 \end{aligned}$$

Therefore, $\Delta^k(\alpha^k U_j^k) = \alpha^k k!$

For every positive integer k, proving (c).

3.2.5 Proof of (d)

$$\begin{aligned}
 \Delta(\alpha U_j)^p &= \alpha^p [(j+1)^p - j] \\
 \Rightarrow \alpha^p [(j+1)^p - j] &= \begin{cases} \text{even} - \text{even} = \text{even for even case} \\ \text{odd} - \text{odd} = \text{odd for odd case} \end{cases}
 \end{aligned}$$

So in all cases $\Delta(\alpha U_j)^p$ and $\Delta(\alpha \hat{U}_j + \beta)^p$ are even, proving (d).

3.2.6 Proof of (e)

Consider $\Delta^k(\alpha U_j)^p$ for $k \geq 2, k < p$. If $k = 2$, then the condition $k < p \Rightarrow p \geq 3$.

$$\begin{aligned}
 \Delta^2(\alpha U_j)^p &= \Delta[\Delta(\alpha U_j)^p] \\
 &= \alpha^p \Delta \left[\sum_{i=1}^{p-1} \binom{p}{i} j^i \right] = \Delta(\text{an even}) \text{ from (a) and (b)} \\
 &= \alpha^p \Delta[(j+1)^p - j^p] = [\Delta[(j+1)^p] - \Delta[j^p]] \alpha^p \\
 &\quad (\text{by the reproductive property.}) \\
 &= \alpha^p [(j+2)^p - (j+1) - [(j+1)^p - j^p]] \\
 &= \alpha^p [(j+2)^p - 2(j+1)^p - j^p] = \begin{cases} \text{even} - \text{even} + \text{even} = \text{even (for even case)} \\ \text{even} - \text{even} + \text{even} = \text{even (for odd case)} \end{cases}
 \end{aligned}$$

So in all cases, $\Delta^2(\alpha U_j)^p$ and $\Delta^2(\alpha \hat{U}_j + \beta)^p$ is even for $p \geq 2$.

Assume the validity of (e) for all positive integer k and p such that $k + p \leq m$, for some positive integer m .

Then

$$\begin{aligned} \Delta^{k+1}(\alpha U_j)^p &= \Delta^k(\Delta(\alpha U_j)^p) \\ &= \Delta^k \alpha^p [(j+2)^p - 2(j+1)^p + j^p] \\ &= \Delta^k \alpha^p [(\Delta^k(j+2)^p) - 2\Delta^k(j+1)^p + \Delta^k(j^p)] \\ &\quad \text{(by the reproductive property of } \Delta^k) \\ &= (\Delta^k(U_{j+2}^p) - 2\Delta^k(U_{j+1}^p) + \Delta^k(U_j^p)) \alpha^p \\ &= \text{even} - \text{even} + \text{even} = \text{even}. \\ &= \alpha^p \Delta^k [U_{j+2}^p] - 2\alpha^p \Delta^k [U_{j+1}^p] + \alpha^p \Delta^k [U_j^p] \end{aligned}$$

Finally, we examine:

$$\begin{aligned} &\Delta^k(\alpha U_j)^{p+1} \\ \Delta^k(\alpha U_j)^{p+1} &= \Delta^k(\alpha j)^{p+1} = \Delta^{k-1}[\Delta(\alpha j)^{p+1}] \\ &= \Delta^{k-1}[\alpha^{p+1} (j+1)^{p+1} - 2(j+1)^{p+1} + j^{p+1}] \\ &= \Delta^{k-1} \left[\alpha^{p+1} [U_{j+2}^{p+1}] - 2 \alpha^{p+1} \Delta^{k-1} [U_{j+1}^{p+1}] + \alpha^{p+1} \Delta^{k-1} [U_j^{p+1}] \right] \\ &= \text{even} - \text{even} + \text{even} = \text{even}. \end{aligned}$$

(by the induction hypothesis). Since $k - 1 + p + 1 = k + p \leq m$.

This completes the proof, that is the relation (e) hold for all +ve integers k and p for which $k < p$, $k \geq 2$. Thus, the theorem is established.

3.3 Corollary

$$\Delta^k(\alpha j)^k = \Delta^k(\alpha^k j^k) = \alpha^k \Delta^k j^k = \alpha^k k!$$

For any integer j and for any positive integer k .

In other words, for any set of even and odd integers $(\alpha U_j$ and $\alpha \hat{U}_j + \beta)$, the k^{th} order difference of the k^{th} power of any integers is equal to $2^k k!$

The implication of (c) in theorem 3.2 is the following: “if any number of consecutive even or odd integers are raised to a positive integral power k , then the k^{th} order difference is equal to $2^k k!$ ”

3.4 Remarks

The existence of the triddle between the difference operator Δ and the D operator (differential operator).

$$D^k(x^p) = 0 \text{ if } k > p$$

$$D^k(x^k) = k!$$

$$D^k(\alpha^k x^k) = \alpha^k k!$$

The coefficient of x^{p-k} in $D^k(x^p)$:

$$D^k(x^p) = \frac{p!}{(p-k)!} x^{p-k} \text{ for } k \geq 2, k < p \text{ is even.}$$

$$\Rightarrow \alpha^p D^k(x^p) = \frac{\alpha^p p!}{(p-k)!} x^{p-k}, 2 \leq k < p \text{ is even}$$

IV. CONCLUSION

This article established the structures of finite orders with respect to powers of consecutive elements of even and odd sets. Specifically, the results reveal a similarity between the difference orders and the D operator powers of monomials with positive integral powers as reflected in (a) and (c) of theorem 3.2 and (I) and (II) of remark 3.4.

REFERENCES

1. Chase, P. E. The elements of arithmetic in which decimal and integral arithmetic are combined and taught inductively. U. Hunt and Son 1844.
2. Finite difference: 2015. Retrieved October, 2015. Available at: <http://www.en.wikipedia.org/wiki/finite-difference>
3. McGraw Hill Dictionary of Scientific and technical terms. 6. E. Perfect power. The McGraw Hill Companies Inc, 2003. Retrieved November 23, 2015. Available at: www.encyclopedia2.thefree-dictionary.com/perfect+power
4. Ladan, Umaru Ibrahim Ukwu Chukwuneny and Apine Elijah. An investigation into the order of integral perfect powers. Ibrahim et al: AJOMCOR 10(1):12:18:2016.
5. Cambridge tracts in Mathematics 87. Exponential Diophantine equations, Cambridge University Press 1988: 169-183.
6. Andreescu, T. and Adrica D. Number theory. Birkhauser Boston, a part of springer science + business, Media, LLCl 2009. DOI: 10.1007/611856.2
7. Perfect Powers: 2015. Retrieved November 23, 2015. Available at: <http://en.wikipedia.org/wiki/perfect-power>

8. Perfect Powers of Successive Naturals: Can you always reach a constant difference? 2015 Retrieved November 23, 2015. Available at: <http://mathstachexchange.com/questions/1345205/perfect-powers-of-successive-natural-can-you-always-reach-a-constant-difference>
9. Catalan's Conjecture; 2002, Retrieved November 23, 2015. Available at: <http://en.wikipedia.org/wiki/catalan%27sconjecture>



Scan to know paper details and
author's profile

The Discrete Ordinates and the Riccati Equation Methods in the Estimation of Growth of Solutions of Systems of Two Linear RST Order Ordinary Differential Equations

G. A. Grigorian

ABSTRACT

The method of comparing the solutions of a system of linear equations with solutions of such a system with piecewise constant coefficients (the discrete ordinates method) and Riccati equation method is used for estimating solutions of systems of two first order linear equations. Two principal (essentially different) cases have been considered, for which some explicit estimates in terms of coefficients of linear systems have been obtained. By examples the obtained results are compared with the results obtained by methods of Liapunov, Yu. S. Bogdanov, T. Wazevski, estimates of solutions by logarithmic norm of S. M. Lozinski and the method of freezing.

Keywords: systems of equations with piecewise constant coefficients, the Riccati equation, normal and extremal solutions, main, nonprincipal and ordinary solutions of the system, the theorem of Wazevski.

Classification: MSC Code: 34C10

Language: English



Great Britain
Journals Press

LJP Copyright ID: 925614

Print ISSN: 2631-8490

Online ISSN: 2631-8504

London Journal of Research in Science: Natural & Formal

Volume 24 | Issue 10 | Compilation 1.0



The Discrete Ordinates and the Riccati Equation Methods in the Estimation of Growth of Solutions of Systems of Two Linear RST Order Ordinary Differential Equations

G. A. Grigorian

ABSTRACT

The method of comparing the solutions of a system of linear equations with solutions of such a system with piecewise constant coefficients (the discrete ordinates method) and Riccati equation method is used for estimating solutions of systems of two first order linear equations. Two principal (essentially different) cases have been considered, for which some explicit estimates in terms of coefficients of linear systems have been obtained. By examples the obtained results are compared with the results obtained by methods of Liapunov, Yu. S. Bogdanov, T. Wazewski, estimates of solutions by logarithmic norm of S. M. Lozinski and the method of freezing.

Keywords: systems of equations with piecewise constant coefficients, the Riccati equation, normal and extremal solutions, main, nonprincipal and ordinary solutions of the system, the theorem of Wazewski.

I. INTRODUCTION

Let $a_{jk}(t)$ ($j, k = 1, 2$) be complex-valued continuous functions on the interval $[t_0, +\infty)$. Consider the system

$$\begin{cases} \phi'(t) = a_{11}(t)\phi(t) + a_{12}(t)\psi(t); \\ \psi'(t) = a_{21}(t)\phi(t) + a_{22}(t)\psi(t), \end{cases} \quad (1.1)$$

$t \geq t_0$. Study of the question of stability of solutions of differential equations and systems of differential equations, in particular of system (1.1), is an important problem of qualitative theory of differential equations and many works are devoted to it (see [1], [2] and cited works therein [3 - 10]). The fundamental theorem of R. Bellman (see [2], pp. 168, 169) reduces the study of boundedness of solutions of wide class of nonlinear systems to the study of stability of linear systems of differential equations. Many problems of mechanics, physics and other natural sciences are connected with the study of stability of the linear

systems of differential equations (in particular of the linear differential equations) too (see for example [6,7]). One of ways to study the mentioned above question is the use of different methods of estimations of solutions of systems of being studied equations (see [4]).

In this paper some estimates of solutions of the system (1.1) in terms of its coefficients are obtained. To obtain them it was used the method of approximation of solutions of (1.1) by solutions of system with piecewise constant coefficients (the discrete ordinates method).

Denote: $P(t) \equiv a_{12}(t) \exp\left\{\int_{t_0}^t [a_{22}(\tau) - a_{11}(\tau)] d\tau\right\}$, $Q(t) \equiv a_{21}(t) \exp\left\{\int_{t_0}^t [a_{11}(\tau) - a_{22}(\tau)] d\tau\right\}$. In this article we will study the following two principal cases:

- A) $P(t) > 0, \quad Q(t) < 0, \quad t \geq t_0;$
- B) $P(t) > 0, \quad Q(t) > 0, \quad t \geq t_0$

(the case $P(t) < 0, \quad Q(t) > 0, \quad t \geq t_0$, is similar to the case A), and the case $P(t) < 0, \quad Q(t) < 0, \quad t \geq t_0$, is reducible to the case B) by the simple substitution $\phi(t) \rightarrow -\phi(t)$). The case A) can be geometrically interpreted as a case, when the origin of coordinates of phase plane of variables u, v is a "center" or a "focus" and the case B) as a "saddle" with respect to the curves $\{(u(t), v(t))\}, \quad t \geq t_0$, where $\{(u(t), v(t))\}$ are the solutions of the system

$$\begin{cases} u'(t) = P(t)v(t); \\ v'(t) = Q(t)u(t), \end{cases} \tag{1.2}$$

$t \geq t_0$. On examples the obtained results are compared with the results obtained by methods of Liapunov, Yu. S. Bogdanov, T. Wazevski, estimates of solutions by logarithmic norm of S. M. Lozinski and of freezing.

II. AUXILIARY PROPOSITIONS

Lemma 2.1. *For each solution $(\phi(t), \psi(t))$ of the system (1.1) and for each $\varepsilon > 0$ and $t_1 > t_0$ there exists piecewise constant functions $\tilde{a}_{jk}, \quad t \geq t_0, \quad j, k = 1, 2$, such, that the solutions $(\tilde{\phi}(t), \tilde{\psi}(t))$ of the system*

$$\begin{cases} \phi'(t) = \tilde{a}_{11}(t)\phi(t) + \tilde{a}_{12}(t)\psi(t); \\ \psi'(t) = \tilde{a}_{21}(t)\phi(t) + \tilde{a}_{22}(t)\psi(t), \end{cases} \tag{2.0}$$

$t \geq t_0$. with $\tilde{\phi}(t_0) = \phi(t_0)$, $\tilde{\psi}(t_0) = \psi(t_0)$ satisfy the inequalities: $|\tilde{\phi}(t_1) - \phi(t_1)| \leq \varepsilon$, $|\tilde{\psi}(t_1) - \psi(t_1)| \leq \varepsilon$.

The proof of this lemma is not difficult, and we omit it.

Remark 2.1. By a solution of the system (2.0) we will mean a pair of absolutely continuous functions $\phi(t)$ and $\psi(t)$, satisfying (2.0) almost everywhere on $[t_0, +\infty)$.

Let $t_0 < t_1 < \dots < t_n < \dots$ be a finite or infinite sequens, and let $\tilde{p}(t) = p_j > 0$, $\tilde{q}(t) = q_j > 0$, $t \in [t_j; t_{j+1})$, $j = 0, 1, 2, \dots$. Consider the Cauchy problem

$$\begin{cases} u'(t) = \tilde{p}(t)v(t); \\ v'(t) = -\tilde{q}(t)u(t); \\ u(t_0) = u_{(0)}, \quad v(t_0) = v_{(0)}, \end{cases} \quad (2.1)$$

$t \geq t_0$. Any solution of this system we will seek in the form

$$u(t) = A_j \sin(\sqrt{l_j}t + \omega_j), \quad v(t) = A_j \sqrt{h_j} \cos(\sqrt{l_j}t + \omega_j), \quad t \in [t_j, t_{j+1}), \quad (2.2)$$

where $l_j \equiv p_j q_j$, $h_j \equiv \frac{q_j}{p_j}$ and A_j, ω_j are the sought constants, $j = 0, 1, 2, \dots$. By virtue of initial conditions of problem (2.1) the unknowns A_0 and ω_0 we determine by solving the system

$$\begin{cases} A_0 \sin(\sqrt{l_0}t_0 + \omega_0) = u_{(0)}; \\ A_0 \sqrt{h_0} \cos(\sqrt{l_0}t_0 + \omega_0) = v_{(0)}, \end{cases} \quad (2.3)$$

and the remaining unknowns by successive solving of the systems

$$\begin{cases} A_{j+1} \sin(\sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}) = A_j \sin(\sqrt{l_j}t_{j+1} + \omega_j); \\ A_{j+1} \sqrt{h_{j+1}} \cos(\sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}) = A_j \sqrt{h_j} \cos(\sqrt{l_j}t_{j+1} + \omega_j), \end{cases} \quad (2.4)$$

$j = 1, 2, \dots$. From (2.3) it follows

$$A_0 = u_{(0)}^2 + \frac{v_{(0)}^2}{h_0}. \quad (2.5)$$

Denote: $\alpha_j \equiv \sqrt{l_j}t_{j+1} + \omega_j$, $\beta_j \equiv \sqrt{l_{j+1}}t_{j+1} + \omega_{j+1}$, $j = 0, 1, 2, \dots$. From (2.4) it is easy to derive the equalities:

$$A_{j+1}^2 = A_j^2 \left[\frac{h_j + h_{j+1}}{2h_{j+1}} + \frac{h_j - h_{j+1}}{2h_{j+1}} \cos 2\alpha_j \right], \quad A_j^2 = A_{j+1}^2 \left[\frac{h_j + h_{j+1}}{2h_j} + \frac{h_{j+1} - h_j}{2h_j} \cos 2\beta_j \right],$$

$j = 0, 1, 2, \dots$. From here it follows

$$|A_{j+1}| \leq |A_j| \leq \sqrt{\frac{h_{j+1}}{h_j}} |A_{j+1}| \quad \text{for} \quad h_j \leq h_{j+1}; \quad (2.6)$$

$$|A_j| \leq |A_{j+1}| \leq \sqrt{\frac{h_j}{h_{j+1}}} |A_j| \quad \text{for} \quad h_j \geq h_{j+1}; \quad (2.7)$$

$j = 0, 1, 2, \dots$. From (2.2) it follows:

$$u^2(t) \frac{1}{h_j} + v^2(t) = A_j^2, \quad t \in [t_j, t_{j+1}), \quad j = 0, 1, 2, \dots \quad (2.8)$$

Let the initial values $u(0)$ and $v(0)$ be real. Then $u(t)$ and $v(t)$ will be real valued. Therefore, from (2.8) we will have:

$$\min\{1, h_j^1\} A_j^2 \leq u^2(t) + v^2(t) \leq \max\{1, h_j^1\} A_j^2, \quad t \in [t_j, t_{j+1}), \quad (2.9)$$

where $h_j^1 \equiv \frac{p_j}{q_j}$, $j = 0, 1, 2, \dots$

Definition 2.1. We shall say, that a continuous on the interval $[t_0; +\infty)$ function $f(t)$ belongs to the class $C_\sim = C_\sim[t_0, +\infty)$, if there exists an infinitely large sequence $\xi_0 = t_0 \leq \xi_1 \leq \dots \leq \xi_n \leq \dots$ such, that $f(t)$ is a nondecreasing on the interval $[\xi_{2n}, \xi_{2n+1}]$ and nonincreasing on the interval $[\xi_{2n+1}, \xi_{2n+2}]$ function, $n = 0, 1, 2, \dots$. The numbers ξ_n , $n = 0, 1, 2, \dots$ we shall call points of possible extremums of the function $f(t)$.

Let $S(t) \in C_\sim$, and let ξ_n , $n = 0, 1, 2, \dots$ be the points of possible extremums of $S(t)$. Note, that if $\xi_{2n+1} = \xi_{2n+2}$, $n = 0, 1, 2, \dots$, then $S(t)$ is a nondecreasing function and if $\xi_{2n} = \xi_{2n+1}$, $n = 0, 1, 2, \dots$, then $S(t)$ is a nonincreasing function. Let $S(t) > 0$, $t \geq t_0$. Consider the functions

$$r_S^-(t) \equiv \begin{cases} 1, & t \in [\xi_0; \xi_1]; \\ \sqrt{\frac{S(\xi_1)}{S(t)}}, & t \in [\xi_1; \xi_2]; \\ \prod_{k=1}^n \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}}, & t \in [\xi_{2n}; \xi_{2n+1}], \quad n = 1, 2, \dots; \\ \left[\prod_{k=1}^{n-1} \sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}} \right] \sqrt{\frac{S(\xi_{2n-1})}{S(t)}}, & t \in [\xi_{2n-1}; \xi_{2n}], \quad n = 2, 3, \dots, \end{cases}$$

$$r_S^+(t) \equiv \begin{cases} \sqrt{\frac{S(t)}{S(\xi_0)}}, & t \in [\xi_0; \xi_1]; \\ \prod_{k=0}^n \sqrt{\frac{S(\xi_{2k+1})}{S(\xi_{2k})}}, & t \in [\xi_{2n+1}; \xi_{2n+2}], \quad n = 0, 1, \dots; \\ \left[\prod_{k=0}^{n-1} \sqrt{\frac{S(\xi_{2k+1})}{S(\xi_{2k})}} \right] \sqrt{\frac{S(t)}{S(\xi_{2n})}}, & t \in [\xi_{2n}; \xi_{2n+1}], \quad n = 1, 2, \dots, \end{cases}$$

Let $S(t)$ be absolutely continuous. Then $\sqrt{\frac{S(\xi_{2n+1})}{S(t)}} = \exp\left\{-\frac{1}{2} \int_{\xi_{2n+1}}^t \frac{S'(\tau)}{S(\tau)} d\tau\right\}$, $n = 0, 1, 2, \dots$,

$\sqrt{\frac{S(\xi_{2k-1})}{S(\xi_{2k})}} = \exp\left\{-\frac{1}{2} \int_{\xi_{2k-1}}^{\xi_{2k}} \frac{S'(\tau)}{S(\tau)} d\tau\right\}$, $k = 1, 2, \dots$. Therefore

$$r_S^-(t) = \exp\left\{\frac{1}{2} \int_{t_0}^t \frac{S'_{(-)}(\tau)}{S(\tau)} d\tau\right\}, \quad t \geq t_0, \tag{2.10}$$

where $S_{(-)}(t) \equiv \begin{cases} -S(t), & \text{if exists } S'(t) > 0; \\ 0, & \text{otherwise,} \end{cases} \quad t \geq t_0$. By analogy it shows, that

$$r_S^+(t) = \exp\left\{\frac{1}{2} \int_{t_0}^t \frac{S'_{(+)}(\tau)}{S(\tau)} d\tau\right\}, \quad t \geq t_0, \tag{2.11}$$

where $S_{(+)}(t) \equiv \begin{cases} S(t), & \text{if exists } S'(t) > 0; \\ 0, & \text{otherwise,} \end{cases} \quad t \geq t_0$. It is clear that $S'(t) = S'_{(+)}(t) - S'_{(-)}(t)$, $|S'(t)| = S'_{(+)}(t) + S'_{(-)}(t)$ in all the points of existence of $S'(t)$. From here, from (2.10) and (2.11) it follows:

$$r_S^-(t) = \sqrt[4]{\frac{S(t_0)}{S(t)}} \exp\left\{\frac{1}{4} \int_{t_0}^t \frac{|S'(\tau)|}{S(\tau)} d\tau\right\}, \quad t \geq t_0, \tag{2.12}$$

$$r_S^+(t) = \sqrt[4]{\frac{S(t)}{S(t_0)}} \exp\left\{\frac{1}{4} \int_{t_0}^t \frac{|S'(\tau)|}{S(\tau)} d\tau\right\}, \quad t \geq t_0, \tag{2.13}$$

Let $p(t)$ and $q(t)$ be positive and continuous functions on the interval $[t_0; +\infty)$. Consider the system of equations

$$\begin{cases} u'(t) = p(t)v(t); \\ v'(t) = -q(t)u(t), \end{cases} \tag{2.14}$$

$t \geq t_0$. Let us introduce some notations, necessary in the sequel: $h(t) \equiv \frac{q(t)}{p(t)}$, $h_1(t) \equiv \frac{p(t)}{q(t)}$, $g(t) \equiv \min\{\sqrt[4]{h(t)}, \sqrt[4]{h_1(t)}\}$, $G(t) \equiv \max\{\sqrt[4]{h(t)}, \sqrt[4]{h_1(t)}\}$, $\|(x(t), y(t))\| \equiv \sqrt{|x(t)|^2 + |y(t)|^2}$, $r_z(t) \equiv \exp\left\{\frac{1}{4} \int_{t_0}^t \frac{|z'(\tau)|}{z(\tau)} d\tau\right\}$, $t \geq t_0$, where $x(t)$ and $y(t)$ are continu-

ous functions on the interval $[t_0; +\infty)$, $z(t)$ is a absolutely continuous and positive function on the interval $[t_0; +\infty)$ with locally finite variation.

Lemma 2.2. *Let $h(t)$ be an absolutely continuous function with locally finite variation. Then for every solution $(u(t), v(t))$ of the system (2.14) the following inequalities hold:*

$$\frac{g(t_0)g(t)}{r_h(t)} \|(u(t_0), v(t_0))\| \leq \|(u(t), v(t))\| \leq G(t_0)G(t) \|(u(t_0), v(t_0))\| r_h(t), \tag{2.15}$$

$t \geq t_0$.

Proof. Let us consider first the case, when $h(t)$ has the additional property: $h(t) \in C_\sim$. Let $(u_0(t), v_0(t))$ be a nontrivial real valued solution of the system (2.14), and let $\xi_0 = t_0 \leq \xi_1 \leq \dots \leq \xi_n \leq \dots$ be the possible extremums of the function $h(t)$. Let $t_1 (> t_0)$ and $\varepsilon (> 0)$ be fixed. By virtue of Lemma 2.1 there exist piecewise constant functions $\tilde{p}(t)$ and $\tilde{q}(t)$ such, that the solution $(\tilde{u}(t), \tilde{v}(t))$ of the system

$$\begin{cases} u'(t) = \tilde{p}(t)v(t); \\ v'(t) = -\tilde{q}(t)u(t), \end{cases}$$

$t \geq t_0$, with $\tilde{u}(t_0) = u_0(t_0)$, $\tilde{v}(t_0) = v_0(t_0)$ satisfies the inequalities:

$$|\tilde{u}(t_1) - u_0(t_1)| \leq \varepsilon, \quad |\tilde{v}(t_1) - v_0(t_1)| \leq \varepsilon. \tag{2.16}$$

Without loss of generality we will assume that $\tilde{p}(\xi_k) = p(\xi_k)$, $\tilde{q}(\xi_k) = q(\xi_k)$, $k = 0, 1, \dots$, $\tilde{p}(t_1) = p(t_1)$, $\tilde{q}(t_1) = q(t_1)$; $\tilde{p}(t) > 0$, $\tilde{q}(t) > 0$, $t \geq t_0$; the function $\frac{\tilde{q}(t)}{\tilde{p}(t)}$ is nondecreasing on the intervals $[\xi_{2k}, \xi_{2k+1}]$ and nonincreasing on the intervals $[\xi_{2k+1}, \xi_{2k+2}]$, $k = 0, 1, 2, \dots$. Then by (2.5) - (2.7), (2.9) the following inequalities hold

$$g_1(t_0)g_1(t_1) \|(u_0(t_0), v_0(t_0))\| \sqrt{\frac{h(\xi_0)}{h(t_1)}} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq G_1(t_0)G_1(t_1) \|(u_0(t_0), v_0(t_0))\|,$$

if $t_1 \in [\xi_0, \xi_1]$;

$$g_1(t_0)g_1(t_1)\|(u_0(t_0), v_0(t_0))\| \sqrt{\frac{h(\xi_0)}{h(t_1)}} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \\ \leq G_1(t_0)G_1(t_1)\|(u_0(t_0), v_0(t_0))\| \sqrt{\frac{h(\xi_1)}{h(t_1)}}, \text{ if } t_1 \in [\xi_1, \xi_2];$$

$$g_1(t_0)g_1(t_1)\|(u_0(t_0), v_0(t_0))\| \left[\prod_{k=0}^n \sqrt{\frac{h(\xi_{2k})}{h(\xi_{2k+1})}} \right] \sqrt{\frac{h(\xi_{2n})}{h(t_1)}} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \\ \leq G_1(t_0)G_1(t_1)\|(u_0(t_0), v_0(t_0))\| \prod_{k=1}^n \sqrt{\frac{h(\xi_{2k-1})}{h(\xi_{2k})}}, \text{ if } t_1 \in [\xi_{2n}, \xi_{2n+1}], \quad n = 1, 2, \dots;$$

$$g_1(t_0)g_1(t_1)\|(u_0(t_0), v_0(t_0))\| \prod_{k=0}^n \sqrt{\frac{h(\xi_{2k})}{h(\xi_{2k+1})}} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \\ \leq G_1(t_0)G_1(t_1)\|(u_0(t_0), v_0(t_0))\| \left[\prod_{k=0}^{n-1} \sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}} \right] \sqrt{\frac{h(\xi_{2n+1})}{h(t_1)}}, \text{ if } t_1 \in [\xi_{2n+1}, \xi_{2n+2}],$$

$n = 1, 2, \dots$, where $g_1(t) \equiv \min\{1, \sqrt{h_1(t)}\}$, $G_1(t) \equiv \max\{1, \sqrt{h_1(t)}\}$, $t \geq t_0$. It follows from here, that

$$\frac{g_1(t_0)g_1(t_1)\|(u_0(t_0), v_0(t_0))\|}{r_h^+(t_1)} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \\ \leq G_1(t_0)G_1(t_1)\|(u_0(t_0), v_0(t_0))\| r_h^-(t_1). \quad (2.17)$$

By analogy (making the substitution $u(t) \rightarrow -u(t)$, interchanging $p(t)$ and $q(t)$, as well as interchanging $u(t)$ and $v(t)$) we come to the inequalities

$$\frac{g_2(t_0)g_2(t_1)\|(u_0(t_0), v_0(t_0))\|}{r_{h_1}^+(t_1)} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \\ \leq G_2(t_0)G_2(t_1)\|(u_0(t_0), v_0(t_0))\| r_{h_1}^-(t_1),$$

where $g_2(t) \equiv \min\{1, \sqrt{h(t)}\}$, $G_2(t) \equiv \max\{1, \sqrt{h(t)}\}$, $t \geq t_0$. From here and from (2.17) we obtain:

$$\sqrt{\frac{g_1(t_0)g_1(t_1)g_2(t_0)g_2(t_1)}{r_h^+(t_1)r_{h_1}^+(t_1)}}\|(u_0(t_0), v_0(t_0))\| \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq$$

$$\leq \sqrt{G_1(t_0)G_1(t_1)G_2(t_0)G_2(t_1)}\|(u_0(t_0), v_0(t_0))\|\sqrt{r_h^-(t_1)r_{h_1}^-(t_1)}. \quad (2.18)$$

Obviously,

$$g_1(t)g_2(t) = g^2(t), \quad G_1(t)G_2(t) = G^2(t), \quad t \geq t_0. \quad (2.19)$$

Note (due to (2.12) and (2.13)), that $r_{h_1}^\pm(t) = r_h^\mp(t)$, $t \geq t_0$. Then $r_{h_1}^\pm(t_1)r_h^\mp(t_1) = r_h^2(t_1)$. From here, from (2.18) and (2.19) we obtain:

$$\begin{aligned} \frac{g(t_0)g(t_1)\|(u_0(t_0), v_0(t_0))\|}{r_h(t_1)} &\leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \\ &\leq G(t_0)G(t_1)\|(u_0(t_0), v_0(t_0))\|r_h(t_1). \end{aligned}$$

From here and from (2.16) it follows:

$$\begin{aligned} \frac{g(t_0)g(t_1)\|(u_0(t_0), v_0(t_0))\|}{r_h(t_1)} - \sqrt{2}\varepsilon &\leq \|(u_0(t_1), v_0(t_1))\| \leq \\ &\leq G(t_0)G(t_1)\|(u_0(t_0), v_0(t_0))\|r_h(t_1) + \sqrt{2}\varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon (> 0)$ from here we will have:

$$\begin{aligned} \frac{g(t_0)g(t_1)\|(u_0(t_0), v_0(t_0))\|}{r_h(t_1)} &\leq \|(u_0(t_1), v_0(t_1))\| \leq \\ &\leq G(t_0)G(t_1)\|(u_0(t_0), v_0(t_0))\|r_h(t_1). \quad (2.20) \end{aligned}$$

Let $(u(t), v(t))$ be an arbitrary (complex) solution of the system (2.14). Since $(u(t), v(t)) = (u_1(t), v_1(t)) + i(u_2(t), v_2(t))$, where $(u_j(t), v_j(t))$, $(j = 1, 2)$ are some real solutions of the system (2.14), by (2.20) we will get:

$$\begin{aligned} \left[\frac{g(t_0)g(t_1)}{r_h(t_1)} \right]^2 \sum_{j=1}^2 \|(u_j(t_0), v_j(t_0))\|^2 &\leq \sum_{j=1}^2 \|(u_j(t_1), v_j(t_1))\|^2 \leq \\ &\leq [G(t_0)G(t_1)r_h(t_1)]^2 \sum_{j=1}^2 \|(u_j(t_0), v_j(t_0))\|^2. \end{aligned}$$

Taking into account the equality $\|(u(t), v(t))\|^2 = \sum_{j=1}^2 \|(u_j(t), v_j(t))\|^2$, $t \geq t_0$, from here we will have:

$$\frac{g(t_0)g(t_1)\|(u(t_0), v(t_0))\|}{r_h(t_1)} \leq \|(u(t_1), v(t_1))\| \leq G(t_0)G(t_1)\|(u(t_0), v(t_0))\|r_h(t_1).$$

By virtue of arbitrariness of $t_1 (> t_0)$ from here it follows (2.15). Thus, we have proved (2.15) under the additional assumption $h(t) \in C_\sim$. Let us prove it in the general case. Let $A_C[t_0; T]$ be the space of absolutely continuous functions $f(t)$ of finite variation on the interval $[t_0; T]$ with the norm $\|f(t)\| \equiv \max_{t \in [t_0; T]} |f(t)| + \int_{t_0}^T |f'(\tau)| d\tau$. Obviously the set of rational functions is everywhere dense in $A_C[t_0, T]$. In view of this we choose polynomials $p_1(t)$ and $q_1(t)$ such, that $p_1(t) > 0$, $q_1(t) > 0$, $t \geq t_0$, and such, that for each fixed $t_1 \in (t_0, T]$ and $\varepsilon (> 0)$ the following inequalities hold

$$\left| \frac{g(t_0)g(t_1)}{r_h(t_1)} - \frac{\tilde{g}(t_0)\tilde{g}(t_1)}{r_{\tilde{h}}(t_1)} \right| \leq \varepsilon, \quad \left| G(t_0)G(t_1)r_h(t_1) - \tilde{G}(t_0)\tilde{G}(t_1)r_{\tilde{h}}(t_1) \right| \leq \varepsilon, \quad (2.21)$$

where $\tilde{g}(t) \equiv \min \left\{ \sqrt[4]{\tilde{h}(t)}, \sqrt[4]{\tilde{h}_1(t)} \right\}$, $\tilde{G}(t) \equiv \max \left\{ \sqrt[4]{\tilde{h}(t)}, \sqrt[4]{\tilde{h}_1(t)} \right\}$, $\tilde{h}(t) \equiv \frac{q_1(t)}{p_1(t)}$, $\tilde{h}_1(t) \equiv \frac{p_1(t)}{q_1(t)}$, $t \geq t_0$, as well as (by Lemma 2.1) the following inequalities hold

$$|u(t_1) - \tilde{u}(t_1)| \leq \varepsilon, \quad |v(t_1) - \tilde{v}(t_1)| \leq \varepsilon, \quad (2.22)$$

where $(\tilde{u}(t), \tilde{v}(t))$ is the solution of the system

$$\begin{cases} u'(t) = & p_1(t)v(t); \\ v'(t) = q_1(t)u(t), \end{cases}$$

$t \geq t_0$, with $\tilde{u}(t_0) = u(t_0)$, $\tilde{v}(t_0) = v(t_0)$. Since obviously $\tilde{h}(t)|_{[t_0; +\infty)} \in C_\sim$, by already proven

$$\frac{\tilde{g}(t_0)\tilde{g}(t_1)\|(u(t_0), v(t_0))\|}{r_{\tilde{h}}(t_1)} \leq \|(\tilde{u}(t_1), \tilde{v}(t_1))\| \leq \tilde{G}(t_0)\tilde{G}(t_1)\|(u(t_0), v(t_0))\|r_{\tilde{h}}(t_1).$$

From here, from (2.21) and (2.22) it follows

$$\begin{aligned} \frac{g(t_0)g(t_1)\|(u(t_0), v(t_0))\|}{r_h(t_1)} - [\sqrt{2} + \|(u(t_0), v(t_0))\|]\varepsilon &\leq \|(u(t_1), v(t_1))\| \leq \\ &\leq G(t_0)G(t_1)\|(u(t_0), v(t_0))\|r_h(t_1) + [\sqrt{2} + \|(u(t_0), v(t_0))\|]\varepsilon. \end{aligned}$$

By virtue of arbitrariness of $t_1 \in (t_0; T]$, $T (> t_0)$ and $\varepsilon (> 0)$ from here it follows (2.15). The lemma is proved.

Let us consider the Cauchy problem

$$\begin{cases} u'(t) = \tilde{p}(t)v(t); \\ v'(t) = \tilde{q}(t)u(t); \\ u(t_0) = u_{(0)}, \quad v(t_0) = v_{(0)}, \end{cases} \quad (2.23)$$

$t \geq t_0$. Its solution we will seek in the form

$$\begin{cases} u(t) = A_j \sqrt{p_j} \exp\{\sqrt{l_j}t\} + B_j \sqrt{p_j} \exp\{-\sqrt{l_j}t\}; \\ v(t) = A_j \sqrt{q_j} \exp\{\sqrt{l_j}t\} - B_j \sqrt{q_j} \exp\{-\sqrt{l_j}t\}, \end{cases} \quad (2.24)$$

$t \in [t_j; t_{j+1})$, $j = 0, 1, 2, \dots$. The unknowns A_0 and B_0 we can find by solving the system

$$\begin{cases} A_0 \sqrt{p_0} \exp\{\sqrt{l_0}t_0\} + B_0 \sqrt{p_0} \exp\{-\sqrt{l_0}t_0\} = u_{(0)}; \\ A_0 \sqrt{q_0} \exp\{\sqrt{l_0}t_0\} - B_0 \sqrt{q_0} \exp\{-\sqrt{l_0}t_0\} = v_{(0)}, \end{cases} \quad (2.25)$$

and the remaining unknowns $A_j, B_j (j = 1, 2, \dots)$ we can find by successive solving the systems

$$\begin{cases} A_j \sqrt{p_j} \exp\{\sqrt{l_j}t_j\} + B_j \sqrt{p_j} \exp\{-\sqrt{l_j}t_j\} = u(t_j); \\ A_j \sqrt{q_j} \exp\{\sqrt{l_j}t_j\} - B_j \sqrt{q_j} \exp\{-\sqrt{l_j}t_j\} = v(t_j), \end{cases}$$

$j = 1, 2, \dots$. We have:

$$A_j = \frac{u(t_j)\sqrt{q_j} + v(t_j)\sqrt{p_j}}{2\sqrt{l_j}} \exp\{-\sqrt{l_j}t_j\}, \quad B_j = \frac{u(t_j)\sqrt{q_j} - v(t_j)\sqrt{p_j}}{2\sqrt{l_j}} \exp\{-\sqrt{l_j}t_j\}.$$

From here, from (2.24) and (2.25) it follows

$$\begin{cases} u(t) = u(t_j) \operatorname{ch}\{\sqrt{l_j}(t - t_j)\} + v(t_j) \sqrt{\frac{p_j}{q_j}} \operatorname{sh}\{\sqrt{l_j}(t - t_j)\}; \\ v(t) = v(t_j) \operatorname{ch}\{\sqrt{l_j}(t - t_j)\} + u(t_j) \sqrt{\frac{q_j}{p_j}} \operatorname{sh}\{\sqrt{l_j}(t - t_j)\}, \end{cases} \quad (2.26)$$

$t \in [t_j; t_{j+1})$, $j = 0, 1, 2, \dots$. From here it is easy to derive the equalities

$$\sqrt{q_j}u(t) + \sqrt{p_j}v(t) = [\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)] \exp\{\sqrt{l_j}(t - t_j)\}, \quad t \in [t_j; t_{j+1}), \quad (2.27)$$

$j = 0, 1, \dots$. From here it follows

$$\begin{aligned} \sqrt{q_{j+1}}u(t_{j+1}) + \sqrt{p_{j+1}}v(t_{j+1}) &= \sqrt{\frac{q_{j+1}}{q_j}} \left[\left(\sqrt{q_j}u(t_j) + \sqrt{\frac{h_j}{h_{j+1}}} \sqrt{p_j}v(t_j) \right) \times \right. \\ &\quad \left. \times \operatorname{ch}\{\sqrt{l_j}(t_{j+1} - t_j)\} + \left(\sqrt{\frac{h_j}{h_{j+1}}} \sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j) \right) \operatorname{sh}\{\sqrt{l_j}(t_{j+1} - t_j)\} \right], \end{aligned} \quad (2.28)$$

$j = 0, 1, 2, \dots$. Let $u(t_0) \geq 0$, $v(t_0) \geq 0$, $u(t_0)^2 + v(t_0)^2 \neq 0$. Then from (2.26) it follows that

$$u(t) > 0, \quad v(t) > 0, \quad t > t_0. \quad (2.29)$$

From here, from (2.27) and (2.28) we get

$$\begin{aligned} \sqrt{\frac{q_{j+1}}{q_j}} [\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)] \exp\{\sqrt{l_j}(t_{j+1} - t_j)\} &\leq \sqrt{q_{j+1}}u(t_{j+1}) + \sqrt{p_{j+1}}v(t_{j+1}) \leq \\ &\leq \sqrt{\frac{p_{j+1}}{p_j}} [\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)] \exp\{\sqrt{l_j}(t_{j+1} - t_j)\}, \end{aligned} \quad (2.30)$$

for $h_j \geq h_{j+1}$, $j = 0, 1, \dots$ and

$$\begin{aligned} \sqrt{\frac{p_{j+1}}{p_j}} [\sqrt{q_j}u(t_j) + \sqrt{p_j}v(t_j)] \exp\{\sqrt{l_j}(t_{j+1} - t_j)\} &\leq \sqrt{q_{j+1}}u(t_{j+1}) + \sqrt{p_{j+1}}v(t_{j+1}) \leq \\ &\leq \sqrt{\frac{q_{j+1}}{q_j}} [\sqrt{q_j}u(t_j) + \sqrt{q_j}v(t_j)] \exp\{\sqrt{l_j}(t_{j+1} - t_j)\}, \end{aligned} \quad (2.31)$$

for $h_j \leq h_{j+1}$, $j = 0, 1, \dots$. Let us consider the system

$$\begin{cases} u'(t) = & p(t)v(t); \\ v'(t) = q(t)u(t), \end{cases} \quad (2.32)$$

$t \geq t_0$. Denote: $e(t) \equiv \exp\left\{\int_{t_0}^t \sqrt{p(\tau)q(\tau)}d\tau\right\}$, $t \geq t_0$.

Lemma 2.3. *Let $h(t)$ be absolutely continuous and has a local finite variation. Then for each solution $(u(t), v(t))$ of the system (2.32) with $u(t_0) \geq 0$, $v(t_0) \geq 0$ the following inequalities hold*

$$\frac{d(u, v)e(t)\sqrt{p(t)}}{\sqrt{p(t_0)r_h^-(t)}} \leq \sqrt{q(t)}u(t) + \sqrt{p(t)}v(t) \leq \frac{d(u, v)e(t)\sqrt{q(t)}}{\sqrt{q(t_0)}}r_h^-(t), \quad t \geq t_0, \quad (2.33)$$

where $d(u, v) \equiv \sqrt{q(t_0)}u(t_0) + \sqrt{p(t_0)}v(t_0)$.

Proof. We prove the lemma only in the particular case when $h(t) \in C_\sim$. The proof in the general case by analogy of the last part of the proof of Lemma 2.2. Let $\varepsilon (> 0)$ and $t_1 (> t_0)$ be fixed, and let $\xi_0 = t_0 \leq \xi_1 \leq \dots \leq \xi_n \leq \dots$ be the possible extremums of the function $h(t)$. Let $(u(t), v(t))$ be a solution of the system (2.32) with $u(t_0) \geq 0, v(t_0) \geq 0$. By virtue of Lemma 2.1 there exist piecewise constant functions $\tilde{p}(t)$ and $\tilde{q}(t)$ such, that the solution $(\tilde{u}(t), \tilde{v}(t))$ of the system

$$\begin{cases} u'(t) = \tilde{p}(t)v(t); \\ v'(t) = \tilde{q}(t)u(t), \end{cases}$$

$t \geq t_0$, with $\tilde{u}(t_0) = u_0(t_0), \tilde{v}(t_0) = v_0(t_0)$ satisfies the inequalities

$$|u(t_1) - \tilde{u}(t_1)| \leq \frac{\varepsilon}{\sqrt{p(t_1)} + \sqrt{q(t_1)}}, \quad |v(t_1) - \tilde{v}(t_1)| \leq \frac{\varepsilon}{\sqrt{p(t_1)} + \sqrt{q(t_1)}}. \quad (2.34)$$

Without loss of generality we will assume that $\tilde{p}(\xi_k) = p(\xi_k), \tilde{q}(\xi_k) = q(\xi_k), k = 0, 1, \dots, \tilde{p}(t_1) = p(t_1), \tilde{q}(t_1) = q(t_1)$; the function $\frac{\tilde{q}(t)}{\tilde{q}(t)}$ is nondecreasing on the intervals $[\xi_{2k}; \xi_{2k+1}]$ and nonincreasing on the intervals $[\xi_{2k+1}; \xi_{2k+2}], k = 0, 1, \dots$;

$$\left| \frac{d(u, v)\tilde{e}(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} - \frac{d(u, v)e(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)}r_h^-(t_1)} \right| \leq \varepsilon, \quad (2.35)$$

$$\left| \frac{d(u, v)\tilde{e}(t_1)\sqrt{q(t_1)}r_h^-(t_1)}{\sqrt{q(t_0)}} - \frac{d(u, v)e(t_1)\sqrt{q(t_1)}r_h^-(t_1)}{\sqrt{q(t_0)}} \right| \leq \varepsilon, \quad (2.36)$$

where $\tilde{e}(t) \equiv \exp\left\{\int_{t_0}^t \sqrt{\tilde{p}(\tau)\tilde{q}(\tau)}d\tau\right\}, t \geq t_0$. Then by (2.30) and (2.31) we will have:

$$\frac{d(u, v)\tilde{e}(t_1)}{\sqrt{p(t_0)}}\sqrt{p(t_1)} \leq \sqrt{q_1(t)}\tilde{u}(t_1) + \sqrt{p_1(t)}\tilde{v}(t_1) \leq \frac{d(u, v)\tilde{e}(t_1)}{\sqrt{q(t_0)}}\sqrt{q(t_1)}, \text{ if } t_1 \in [\xi_0, \xi_1];$$

$$\frac{d(u, v)\tilde{e}(t_1)}{\sqrt{p(t_0)}}\sqrt{\frac{h(t_1)}{h(\xi_1)}}\sqrt{p(t_1)} \leq \sqrt{q_1(t)}\tilde{u}(t_1) + \sqrt{p_1(t)}\tilde{v}(t_1) \leq \frac{d(u, v)\tilde{e}(t_1)}{\sqrt{q(t_0)}}\sqrt{\frac{h(\xi_1)}{h(t_1)}}\sqrt{q(t_1)},$$

if $t_1 \in [\xi_0, \xi_1]$;

$$\frac{d(u, v)\tilde{e}(t_1)}{\sqrt{p(t_0)}}\left(\prod_{k=0}^{n-1}\sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}}\right)\sqrt{p(t_1)} \leq \sqrt{q_1(t)}\tilde{u}(t_1) + \sqrt{p_1(t)}\tilde{v}(t_1) \leq$$

$$\leq \frac{d(u, v)\tilde{e}(t_1)}{\sqrt{q(t_0)}} \left(\prod_{k=0}^{n-1} \sqrt{\frac{h(\xi_{2k+2})}{h(\xi_{2k+1})}} \right) \sqrt{q(t_1)}, \quad \text{if } t_1 \in [\xi_{2n}; \xi_{2n+1}], \quad n = 1, 2, \dots;$$

$$\begin{aligned} & \frac{d(u, v)\tilde{e}(t_1)}{\sqrt{p(t_0)}} \left(\prod_{k=0}^{n-1} \sqrt{\frac{h(\xi_{2k+1})}{h(\xi_{2k+2})}} \right) \sqrt{\frac{h(t_1)}{h(\xi_{2n+1})}} \sqrt{p(t_1)} \leq \sqrt{q_1(t)}\tilde{u}(t_1) + \sqrt{p_1(t)}\tilde{v}(t_1) \leq \\ & \leq \frac{d(u, v)\tilde{e}(t_1)}{\sqrt{q(t_0)}} \left(\prod_{k=0}^{n-1} \sqrt{\frac{h(\xi_{2k+2})}{h(\xi_{2k+1})}} \right) \sqrt{\frac{h(\xi_{2n+1})}{h(t_1)}} \sqrt{q(t_1)}, \quad \text{if } t_1 \in [\xi_{2n+1}; \xi_{2n+2}], \quad n = 1, 2, \dots; \end{aligned}$$

Therefore

$$\frac{d(u, v)\tilde{e}(t_1)\sqrt{p(t)}}{\sqrt{p(t_0)r_h^-(t_1)}} \leq \sqrt{q(t_1)}\tilde{u}(t_1) + \sqrt{p(t_1)}\tilde{v}(t_1) \leq \frac{d(u, v)\tilde{e}(t_1)\sqrt{q(t_1)}}{q(t_0)}r_h^-(t_1).$$

From here and from (2.34) - (2.36) it follows that

$$\frac{d(u, v)e(t_1)\sqrt{p(t_1)}}{\sqrt{p(t_0)r_h^-(t_1)}} - 2\varepsilon \leq \sqrt{q(t_1)}u(t_1) + \sqrt{p(t_1)}v(t_1) \leq \frac{d(u, v)e(t_1)\sqrt{q(t_1)}}{q(t_0)}r_h^-(t_1) + 2\varepsilon.$$

By virtue of arbitrariness of $\varepsilon (> 0)$ and $t_1 (> t_0)$ from here it follows (2.33). The lemma is proved.

Consider the Riccati equation

$$y'(t) + p(t)y^2(t) - q(t) = 0, \quad t \geq t_0. \quad (2.37)$$

The solutions $y(t)$ of this equation, existing on the interval $[t_1, t_2]$ ($t_0 \leq t_1 < t_2 \leq +\infty$) are connected with the solutions $(u(t), v(t))$ of the system (2.32) by equalities (see [11], pp. 153, 154):

$$u(t) = u(t_1) \exp \left\{ \int_{t_2}^t p(\tau)y(\tau)d\tau \right\}, \quad v(t) = y(t)u(t), \quad t \in [t_1, t_2]. \quad (2.38)$$

Let $y_0(t)$ be a solution of Eq. (2.37) with $y_0(t_0) > 0$. It follows from Theorem 4.1 of work [12] (see [12], p. 26) that $y_0(t)$ exists on the interval $[t_0, +\infty)$ and

$$y_0(t) > 0, \quad t \geq t_0. \quad (2.39)$$

Since $p(t) > 0$, $q(t) > 0$, $t \geq t_0$, then from Theorem 3.1 of work [13] (see [13], p. 4) it follows that

$$y_0(s) > \frac{y_0(t)}{1 + y_0(t) \int_t^s p(\zeta)d\zeta}, \quad s \geq t \geq t_0. \quad (2.40)$$

Consider the integral

$$\nu_{y_0}(t) \equiv \int_t^{+\infty} p(\tau) \exp \left\{ -2 \int_t^\tau p(\xi) y_0(\xi) d\xi \right\} d\tau, \quad t \geq t_0.$$

From (2.39) and (2.40) it follows that

$$\nu_{y_0}(t) \leq \int_t^{+\infty} p(\tau) \exp \left\{ -2 \int_t^\tau \frac{p(s)y_0(t)}{1 + y_0(t) \int_t^s p(\zeta) d\zeta} ds \right\} d\tau = \frac{1}{y_0(t)}, \quad t \geq t_0. \quad (2.41)$$

The function $y_*(t) \equiv y_0(t) - \frac{1}{\nu_{y_0}(t)}$, $t \geq t_0$, is an extremal solution of Eq. (2.37) (see [14], p. 194, Theorem 4). From (2.41) it follows that

$$y_*(t) \leq 0, \quad t \geq t_0. \quad (2.42)$$

Let us show that

$$y_*(t) < 0, \quad t \geq t_0. \quad (2.43)$$

Suppose that $y_*(t_1) = 0$ for some $t_1 \geq t_0$. Then by virtue of Theorem 4.1 of work [13] the following inequality holds $y_*(t) \geq 0$, $t \geq t_1$. From here and from (2.42) it follows that $y_*(t) \equiv 0$ on the interval $[t_1, +\infty)$, which is impossible. The obtained contradiction proves (2.43). Since $y_0(t_0) > 0$, then from (2.42) ((2.43)) and from Theorem 4 of work [14] it follows that $y_0(t)$ is a normal solution (a solution $y(t)$ of Eq. (2.37) is said to be normal if there exists a neighborhood of the point $y(t_0)$ such that every solution of Eq. (2.37) with initial value from this neighborhood exists on the interval $[t_0, +\infty)$). Then (see [14], p. 195)

$$\int_{t_0}^{+\infty} p(\tau)[y_0(\tau) - y_*(\tau)]d\tau = +\infty. \quad (2.44)$$

Definition 2.1. *The solution $(u_*(t), v_*(t))$ of the system (2.32), satisfying the initial conditions $u_*(t_0) = 1$, $v_*(t_0) = y_*(t_0)$, will be called the canonical main solution of Eq. (2.32). The (a) solution $(u_0(t), v_0(t))$ of the system (2.32), satisfying the initial conditions $u_0(t_0) = v_0(t_0) = 1$ ($u_0(t_0) \geq 0$, $v_0(t_0) \geq 0$, $u_0^2(t_0) + v_0^2(t_0) \neq 0$), will be called the canonical nonprincipal (a real nonprincipal) solution of the system (2.32). The solutions $\lambda(u_*(t), v_*(t))$ and $\lambda(u_0(t), v_0(t))$, where λ is an arbitrary constant and $(u_0(t), v_0(t))$ is a real nonprincipal solution of the system (2.32), will be called a main and a principal*

solutions of the system (2.32) respectively. A solution of the system (2.32), which is not main solution will be called an ordinary solution of the system (2.32).

From (2.43) it follows that the canonical main and nonprincipal solutions of the system (2.32) are linearly independent. Therefore for general solution $(u(t), v(t))$ of the system (2.32) the following representation holds

$$(u(t), v(t)) = \lambda_0(u_0(t), v_0(t)) + \lambda_*(u_*(t), v_*(t)), \quad \lambda_0 = \text{const}, \quad \lambda_* = \text{const}, \quad t \geq t_0. \quad (2.45)$$

On the strength of (2.38) we have

$$u_*(t) = \exp\left\{\int_{t_0}^t p(\tau)y_*(\tau)d\tau\right\}, \quad v_*(t) = y_*(t)u_*(t), \quad t \geq t_0; \quad (2.46)$$

$$u_0(t) = \exp\left\{\int_{t_0}^t p(\tau)y_0(\tau)d\tau\right\}, \quad t \geq t_0, \quad (2.47)$$

where $y_0(t)$ is the solution of eq. (2.37) with $y_0(t_0) = 1$. From here and from (2.44) it follows that

$$\frac{u_*(t)}{u_0(t)} \rightarrow 0 \quad \text{for } t \rightarrow +\infty. \quad (2.48)$$

Let us show that

$$\frac{v_*(t)}{v_0(t)} \rightarrow 0 \quad \text{for } t \rightarrow +\infty. \quad (2.49)$$

In Eq. (2.37) we make the change $y(t) = \frac{1}{z(t)}$. We will come to the equation

$$z'(t) + q(t)z^2(t) - p(t) = 0, \quad t \geq t_0. \quad (2.50)$$

To prove (2.49) we need to the following

Lemma 2.4. Suppose $\int_{t_0}^{+\infty} p(t)dt = +\infty$ or $\int_{t_0}^{+\infty} q(t)dt = +\infty$. Then $z_*(t) \equiv \frac{1}{y_*(t)}$, $t \geq t_0$, is the extremal solution of Eq. (2.50), where $y_*(t)$ is the extremal solution of Eq. (2.37).

Proof. Above it was shown that $y_*(t) < 0$, $t \geq t_0$. Therefore $z_*(t)$ is defined correct. Obviously $z_*(t)$ is a solution to Eq. (2.50). Suppose $\int_{t_0}^{+\infty} p(t)dt = +\infty$. Let $\tilde{z}_*(t)$ be the extremal solution of Eq. (2.50). Suppose $z_*(t) \neq \tilde{z}_*(t)$. Then the solution $z_N(t)$ of Eq. (2.50) with $z_N(t_0) = \frac{z_*(t_0) + \tilde{z}_*(t_0)}{2}$ is a normal solution to Eq. (2.50). and $z_N(t) < 0$, $t \geq t_0$ (the graph of $z_N(t)$ is between the graphs of $z_*(t)$ and $\tilde{z}_*(t)$). Therefore $y_N(t) \equiv \frac{1}{z_N(t)}$, $t \geq$

t_0 is a normal solution of Eq. (2.37). Hence $\nu_{y_N}(t_0) < +\infty$. On the other hand since $y_N(t) < 0, t \geq t_0$ we have $\nu_{y_N}(t_0) \geq \int_{t_0}^{+\infty} p(t)dt = +\infty$. The obtained contradiction shows that $z_*(t)$ is extremal. Suppose now $\int_{t_0}^{+\infty} q(t)dt = +\infty$. Let $z_*(t)$ not be extremal. the the integral $\int_{t_0}^{+\infty} q(t) \exp\left\{-\int_{t_0}^t q(s)z_*(s)ds\right\}dt$ is convergent. On the other hand since $z_*(t) < 0, t \geq t_0$, we have $\int_{t_0}^{+\infty} q(t) \exp\left\{-\int_{t_0}^t q(s)z_*(s)ds\right\}dt \geq \int_{t_0}^{+\infty} q(t)dt = +\infty$. The obtained contradiction completes the proof of the lemma.

Obviously $z_0(t) \equiv \frac{1}{y_0(t)}$ is a normal solutions of Eq. (2.50). Then since $z_*(t)$ is extremal we have

$$\int_{t_0}^{+\infty} q(\tau)[z_0(\tau) - z_*(\tau)]d\tau = +\infty. \tag{2.51}$$

Let $\tilde{v}_0(t) \equiv \exp\left\{\int_{t_0}^t q(\tau)z_0(\tau)d\tau\right\}, \tilde{u}_0(t) = z_0(t)\tilde{v}_0(t), \tilde{v}_*(t) \equiv \exp\left\{\int_{t_0}^t q(\tau)z_*(\tau)d\tau\right\},$

$\tilde{u}_*(t) = z_*(t)\tilde{v}_*(t), t \geq t_0$. By virtue of (2.38) $(\tilde{u}_0(t), \tilde{v}_0(t))$ and $(\tilde{u}_*(t), \tilde{v}_*(t))$ are solutions of the system (2.32). From (2.51) it follows that

$$\frac{\tilde{v}_*(t)}{\tilde{v}_0(t)} \rightarrow 0 \text{ for } t \rightarrow +\infty. \tag{2.52}$$

Since $\tilde{u}_0(t_0) = \tilde{v}_0(t_0) = u_0(t_0) = v_0(t_0) = 1, \tilde{u}_*(t_0) = \frac{1}{y_*(t_0)}, \tilde{v}_*(t_0) = 1$, we have $(\tilde{u}_0(t), \tilde{v}_0(t)) = (u_0(t), v_0(t)), (\tilde{u}_*(t), \tilde{v}_*(t)) = \frac{1}{y_*(t_0)}(u_*(t), v_*(t)), t \geq t_0$. From here and from (2.52) it follows (2.49). From (2.44), (2.48) and (2.49) it follows

$$(u(t), v(t)) = \lambda_0(u_0(t), v_0(t))[1 + o(1)], \quad t \rightarrow +\infty. \tag{2.53}$$

By (2.32) from (2.43) and (2.46) we will have

$$0 < u_*(t) \leq u_*(t_0), \quad v_*(t_0) \leq v_*(t) < 0, \quad t \geq t_0. \tag{2.54}$$

III. ESTIMATES OF THE SOLUTIONS OF THE SYSTEM (1.1)

In the system (1.1) we make the substitutions

$$\phi(t) = \exp\left\{\int_{t_0}^t a_{11}(\tau)d\tau\right\}u(t), \quad \psi(t) = \exp\left\{\int_{t_0}^t a_{22}(\tau)d\tau\right\}v(t), \quad (3.1)$$

We will come to the system (1.2). In the sequel we will assume that the function $\frac{a_{12}(t)}{a_{21}(t)}$ is absolutely continuous and has a locally finite variation. Denote:

$$m(t) \equiv \min\left\{\sqrt[4]{\left|\frac{a_{12}(t)}{a_{21}(t)}\right|}, \sqrt[4]{\left|\frac{a_{21}(t)}{a_{12}(t)}\right|}\right\}, \quad M(t) \equiv \max\left\{\sqrt[4]{\left|\frac{a_{12}(t)}{a_{21}(t)}\right|}, \sqrt[4]{\left|\frac{a_{21}(t)}{a_{12}(t)}\right|}\right\},$$

$$\mathcal{F}(t) \equiv \left|\int_{t_0}^t [Re a_{11}(\tau) - Re a_{22}(\tau)]d\tau\right| + \int_{t_0}^t \left|\frac{1}{4} \left(\frac{a_{12}(\tau)}{a_{21}(\tau)}\right)' \frac{a_{21}(\tau)}{a_{12}(\tau)} + \frac{a_{22}(\tau) - a_{11}(\tau)}{2}\right|d\tau, \quad t \geq t_0.$$

Theorem 3.1. *Let the condition A) be satisfied. Then for each solution $(\phi(t), \psi(t))$ of the system (1.1) the following inequalities hold*

$$\begin{aligned} m(t_0)\|(\phi(t_0), \psi(t_0))\| & m(t) \exp\left\{\int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 Re a_{jj}(\tau)\right]d\tau - \mathcal{F}(t)\right\} \leq \|(\phi(t), \psi(t))\| \leq \\ & \leq M(t_0)\|(\phi(t_0), \psi(t_0))\| M(t) \exp\left\{\int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 Re a_{jj}(\tau)\right]d\tau + \mathcal{F}(t)\right\}, \quad t \geq t_0. \end{aligned} \quad (3.2)$$

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (1.1), and let $(u(t), v(t))$ be the solution of the system (1.2), satisfying the initial conditions $u(t_0) = \phi(t_0)$, $v(t_0) = \psi(t_0)$. Then by virtue of (3.1) we have

$$|\phi(t)| = \exp\left\{\int_{t_0}^t Re a_{11}(\tau)d\tau\right\}|u(t)|, \quad |\psi(t)| = \exp\left\{\int_{t_0}^t Re a_{22}(\tau)d\tau\right\}|v(t)|, \quad t \geq t_0.$$

From here it follows

$$\|(\phi(t), \psi(t))\| = \sqrt{\exp\left\{2 \int_{t_0}^t Re a_{11}(\tau)d\tau\right\}|u(t)|^2 + \exp\left\{2 \int_{t_0}^t Re a_{22}(\tau)d\tau\right\}|v(t)|^2}, \quad t \geq t_0.$$

Therefore

$$\begin{aligned} \exp\left\{\min\left\{\int_{t_0}^t \operatorname{Re} a_{11}(\tau)d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau)d\tau\right\}\right\} \|(u(t), v(t))\| &\leq \|(\phi(t), \psi(t))\| \leq, \quad t \geq t_0. \\ &\leq \exp\left\{\max\left\{\int_{t_0}^t \operatorname{Re} a_{11}(\tau)d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau)d\tau\right\}\right\} \|(u(t), v(t))\|, \quad t \geq t_0. \end{aligned} \quad (3.3)$$

Denote $H(t) \equiv \left|\frac{Q(t)}{P(t)}\right|$, $H_1(t) \equiv \left|\frac{P(t)}{Q(t)}\right|$, $w(t) \equiv \min\{\sqrt[4]{H(t)}, \sqrt[4]{H_1(t)}\}$, $W(t) \equiv \max\{\sqrt[4]{H(t)}, \sqrt[4]{H_1(t)}\}$, $t \geq t_0$. By virtue of Lemma 2.2 from the condition of the theorem it follows

$$w(t_0)\|(\phi(t_0), \psi(t_0))\| \frac{w(t)}{r_H(t)} \leq \|(u(t), v(t))\| \leq W(t_0)W(t)\|(\phi(t_0), \psi(t_0))\| r_H(t), \quad t \geq t_0.$$

From here and from (3.3) we will get

$$\begin{aligned} w(t_0)\|(\phi(t_0), \psi(t_0))\| \frac{w(t)}{r_H(t)} \exp\left\{\min\left\{\int_{t_0}^t \operatorname{Re} a_{11}(\tau)d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau)d\tau\right\}\right\} &\leq \|(\phi(t), \psi(t))\| \leq \\ &\leq W(t_0)\|(\phi(t_0), \psi(t_0))\| W(t) \exp\left\{\max\left\{\int_{t_0}^t \operatorname{Re} a_{11}(\tau)d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau)d\tau\right\}\right\} r_H(t), \quad t \geq t_0. \end{aligned}$$

Since $w(t_0) = m(t_0)$, $W(t_0) = M(t_0)$,

$$m(t) \exp\left\{\min\left\{\int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau, \int_{t_0}^t \frac{\operatorname{Re} a_{22}(\tau) - \operatorname{Re} a_{11}(\tau)}{2} d\tau\right\}\right\} \leq w(t),$$

$$W(t) \leq M(t) \exp\left\{\max\left\{\int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau, \int_{t_0}^t \frac{\operatorname{Re} a_{22}(\tau) - \operatorname{Re} a_{11}(\tau)}{2} d\tau\right\}\right\}, \quad t \geq t_0,$$

taking into account the equalities

$$\begin{aligned} \min\left\{\int_{t_0}^t \operatorname{Re} a_{11}(\tau)d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau)d\tau\right\} &= \\ &= \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau)\right] d\tau - \left|\int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau\right|, \end{aligned}$$

$$\begin{aligned} \max \left\{ \int_{t_0}^t \operatorname{Re} a_{11}(\tau) d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau) d\tau \right\} &= \\ &= \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) \right] d\tau + \left| \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau \right|, \\ \min \left\{ \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau, \int_{t_0}^t \frac{\operatorname{Re} a_{22}(\tau) - \operatorname{Re} a_{11}(\tau)}{2} d\tau \right\} &= \\ &= - \left| \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau \right|, \\ \max \left\{ \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau, \int_{t_0}^t \frac{\operatorname{Re} a_{22}(\tau) - \operatorname{Re} a_{11}(\tau)}{2} d\tau \right\} &= \\ &= \left| \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau \right|, \end{aligned}$$

$t \geq t_0$. from (3.4) we will get (3.2). The theorem is proved.

Remark 3.1. Let $a(t)$ and $b(t)$ be some continuous functions on $[t_0; +\infty)$ and let $b(t) > 0$, $t \geq t_0$. Consider the system

$$\begin{cases} \phi'(t) = a(t)\phi(t) + b(t)\psi(t); \\ \psi'(t) = -b(t)\phi(t) + a(t)\psi(t), \quad t \geq t_0. \end{cases}$$

For this system we have $\mathcal{F}(t) \equiv 0$, $m(t) = M(t) \equiv 1$. Therefore by Theorem 3.1 for its general solution $(\phi(t), \psi(t))$ the inequalities

$$\|(\phi(t_0), \psi(t_0))\| \exp \left\{ \int_{t_0}^t \operatorname{Re} a(\tau) d\tau \right\} \leq \|(\phi(t), \psi(t))\| \leq \|(\phi(t_0), \psi(t_0))\| \exp \left\{ \int_{t_0}^t \operatorname{Re} a(\tau) d\tau \right\},$$

$t \geq t_0$, are fulfilled. Hence

$$\|(\phi(t), \psi(t))\| = \|(\phi(t_0), \psi(t_0))\| \exp \left\{ \int_{t_0}^t \operatorname{Re} a(\tau) d\tau \right\}, \quad t \geq t_0,$$

and in this sense the estimates (3.2) are sharp.

Example 3.1. Consider the system

$$\begin{cases} \phi'(t) = (-\lambda + \sin t)\phi(t) + t^\alpha \psi(t); \\ \psi'(t) = -t^\beta \phi(t) + (-\mu + \cos t)\psi(t), \end{cases} \quad (3.5)$$

$t \geq \frac{\pi}{4}$, where λ , μ , α and β are some real numbers. For this system $m(t) = t^{-\frac{|\alpha-\beta|}{2}}$, $M(t) = t^{\frac{|\alpha-\beta|}{2}}$, $\mathcal{F}(t) = |\sqrt{2} + \lambda - \mu + (\lambda - \mu)t| + \int_{\pi/4}^t \left| \frac{\alpha-\beta}{4\tau} + \frac{\lambda-\mu}{2} + \frac{\sqrt{2}}{2} \cos(\tau + \frac{\pi}{4}) \right| d\tau$,

$t \geq \frac{\pi}{4}$. Using Theorem 3.1 it is easy to find the following regions of values of the parameters λ , μ , α , β for which the system (3.5) is asymptotically stable:

$$O_1 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + \sqrt{2}, \lambda > 0, \mu > 0\};$$

$$O_2 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda = \mu > \frac{\sqrt{2}}{2\pi}\};$$

$$O_3 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| > 3\sqrt{2}, \lambda > 0, \mu > 0\};$$

$O_3 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| = 3\sqrt{2}, \lambda > 0, \mu > 0, \alpha = \beta\}$; and the following regions of values of parameters λ , μ , α , β for which system (3.5) is unstable:

$$O_5 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| + \sqrt{2} < 0, \lambda < 0, \mu < 0\};$$

$$O_6 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda = \mu < -\frac{\sqrt{2}}{2\pi}\};$$

$$O_7 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| < 0, |\lambda - \mu| > \sqrt{2}, \lambda < 0, \mu < 0\};$$

$$O_7 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| < 0, |\lambda - \mu| = \sqrt{2}, \lambda < 0, \mu < 0, \alpha = \beta\};$$

It is not difficult to verify that the application of the estimates of Liapunov (see [4], p. 432), Yu. S. Bogdanov (see [4], p. 433) and estimate by freezing method (see [4], p. 441) to the system (3.5) give no result. The estimates by logarithmic norms γ_I and γ_{II} of S. M. Lozinski (see [4], pp. 435, 436) give result only for $\lambda > 0$, $\mu > 0$, $\alpha < -1$, $\beta < -1$. For comparison now we use the theorem of Wazewski to the system (3.5) (see [4], p. 434). By virtue of this theorem for each solution $(\phi(t), \psi(t))$ of the system (3.5) the following inequalities hold

$$\begin{aligned} \|(\phi(t), \psi(t))\| \exp\left\{ \int_{\pi/4}^t \omega_-(\tau) d\tau \right\} &\leq \|(\phi(t), \psi(t))\| \leq \\ &\leq \|(\phi(t), \psi(t))\| \exp\left\{ \int_{\pi/4}^t \omega_+(\tau) d\tau \right\}, \quad t \geq \frac{\pi}{4}, \end{aligned} \quad (3.6)$$

where $\omega_{\pm} \equiv \frac{-\lambda - \mu + \sin t + \cos t \pm \sqrt{(\lambda - \mu + \cos t - \sin t)^2 + (t^{\alpha} - t^{\beta})^2}}{2}$. If $\alpha \neq \beta > 0$ or $\beta \neq \alpha > 0$, then from (3.6) does not follow neither asymptotic stability nor instability of the system (3.5) for every values of λ and μ .

Definition 3.1. A solution $(\phi(t), \psi(t))$ of the system (1.1), satisfying the condition B), is said to be a main (a nonprincipal, an ordinary) solution of the system (1.1), if $\phi(t_0) = u(t_0)$, $\psi(t_0) = v(t_0)$, where $(u(t), v(t))$ is a main (a nonprincipal, an ordinary) solution of the system (1.2).

Theorem 3.2. Let the condition B) be satisfied and let

$$C) \int_{t_0}^{+\infty} a_{12}(t) \exp \left\{ \int_{t_0}^t [a_{22}(s) - a_{11}(s)] ds \right\} dt = +\infty \text{ or} \\ \int_{t_0}^{+\infty} a_{21}(t) \exp \left\{ \int_{t_0}^t [a_{11}(s) - a_{22}(s)] ds \right\} dt = +\infty.$$

Then if:

i) $(\phi(t), \psi(t))$ is a nonprincipal solution of the system (1.1), then

$$D(\phi, \psi)m(t) \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) + \sqrt{a_{12}(\tau)a_{21}(\tau)} \right] d\tau - \mathcal{F}(t) \right\} \leq |\phi(t)| + |\psi(t)| \leq \\ \leq D(\phi, \psi)M(t) \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) + \sqrt{a_{12}(\tau)a_{21}(\tau)} \right] d\tau + \mathcal{F}(t) \right\}, \quad \psi t \geq t_0, \quad (3.7)$$

$$\text{where } D(\phi, \psi) \equiv \sqrt[4]{\left| \frac{a_{21}(t_0)}{a_{12}(t_0)} \right|} |\phi(t_0)| + \sqrt[4]{\left| \frac{a_{12}(t_0)}{a_{21}(t_0)} \right|} |\psi(t_0)|;$$

ii) $(\phi(t), \psi(t))$ is a main solution of the system (1.1), then

$$|\phi(t)| + |\psi(t)| \leq \\ \leq (|\phi(t_0)| + |\psi(t_0)|) \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) \right] d\tau + \left| \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau \right| \right\}, \quad (3.8)$$

$t \geq t_0$;

iii) $(\phi(t), \psi(t))$ is an ordinary solution of the system (1.1), then

$$c_1 m(t) \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) + \sqrt{a_{12}(\tau)a_{21}(\tau)} \right] d\tau - \mathcal{F}(t) \right\} \leq |\phi(t)| + |\psi(t)| \leq$$

$$\leq c_2 M(t) \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) + \sqrt{a_{12}(\tau)a_{21}(\tau)} \right] d\tau + \mathcal{F}(t) \right\}, \quad t \geq t_0, \quad (3.9)$$

where $c_j = \text{const} > 0$, $j = 1, 2$.

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (1.1), and $(u(t), v(t))$ be the solution of the system (1.2) with $u(t_0) = \phi(t_0)$, $v(t_0) = \psi(t_0)$. Then by (3.1) we have

$$|\phi(t)| + |\psi(t)| = \exp \left\{ \int_{t_0}^t \operatorname{Re} a_{11}(\tau) d\tau \right\} |u(t)| + \exp \left\{ \int_{t_0}^t \operatorname{Re} a_{22}(\tau) d\tau \right\} |v(t)|, \quad t \geq t_0.$$

From here it follows

$$\begin{aligned} \exp \left\{ \min \left\{ \int_{t_0}^t \operatorname{Re} a_{11}(\tau) d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau) d\tau \right\} \right\} (|u(t)| + |v(t)|) &\leq |\phi(t)| + |\psi(t)| \leq \\ &\leq \exp \left\{ \max \left\{ \int_{t_0}^t \operatorname{Re} a_{11}(\tau) d\tau, \int_{t_0}^t \operatorname{Re} a_{22}(\tau) d\tau \right\} \right\} (|u(t)| + |v(t)|), \quad t \geq t_0, \end{aligned}$$

or, which is the same,

$$\begin{aligned} \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) \right] d\tau - \left| \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau \right| \right\} (|u(t)| + |v(t)|) &\leq \\ &\leq |\phi(t)| + |\psi(t)| \leq \\ &\leq \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 \operatorname{Re} a_{jj}(\tau) \right] d\tau + \left| \int_{t_0}^t \frac{\operatorname{Re} a_{11}(\tau) - \operatorname{Re} a_{22}(\tau)}{2} d\tau \right| \right\} (|u(t)| + |v(t)|), \quad (3.10) \end{aligned}$$

$t \geq t_0$. Let $(u_0(t), v_0(t))$ be a real nonprincipal solution of the system (1.2). Then by virtue of Lemma 2.3 and (2.29) we have

$$\frac{\tilde{D}E(t)\sqrt{P(t)}}{\sqrt{P(t_0)}r_H^-(t)} \leq \sqrt{Q(t)}u_0(t) + \sqrt{P(t)}v_0(t) \leq \frac{\tilde{D}E(t)\sqrt{Q(t)}}{\sqrt{Q(t_0)}}r_H^-(t), \quad t \geq t_0, \quad (3.11)$$

where $\tilde{D} \equiv \sqrt{Q(t_0)}u_0(t) + \sqrt{P(t_0)}v_0(t) = \sqrt{a_{21}(t_0)}\phi(t_0) + \sqrt{a_{12}(t_0)}\psi(t_0)$, $E(t) \equiv$

$$\equiv \exp\left\{\int_{t_0}^t \sqrt{P(\tau)Q(\tau)}d\tau\right\} = \exp\left\{\int_{t_0}^t \sqrt{a_{12}(\tau)a_{21}(\tau)}d\tau\right\}, \quad t \geq t_0. \text{ Obviously}$$

$$\begin{aligned} \min\{\sqrt{P(t)}, \sqrt{Q(t)}\}[u_0(t)+v_0(t)] &\leq \sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t) \leq \\ &\leq \max\{\sqrt{P(t)}, \sqrt{Q(t)}\}[u_0(t) + v_0(t)], \quad t \geq t_0. \end{aligned}$$

Therefore,

$$\begin{aligned} \min\left\{\frac{1}{\sqrt{P(t)}}, \frac{1}{\sqrt{Q(t)}}\right\} \left[\sqrt{Q(t)}u_0(t)+\sqrt{P(t)}v_0(t)\right] &\leq u_0(t)+v_0(t) \leq \\ &\leq \max\left\{\frac{1}{\sqrt{P(t)}}, \frac{1}{\sqrt{Q(t)}}\right\} \left[\sqrt{Q(t)}u_0(t) + \sqrt{P(t)}v_0(t)\right], \quad t \geq t_0. \end{aligned}$$

From here and from (3.11) we will get

$$\begin{aligned} \tilde{D} \min\left\{\sqrt[4]{\frac{P(t)}{Q(t)}}, \sqrt[4]{\frac{Q(t)}{P(t)}}\right\} \frac{E(t)}{\sqrt[4]{P(t_0)Q(t_0)}r_H(t)} &\leq u_0(t)+v_0(t) \leq \\ &\leq \tilde{D} \max\left\{\sqrt[4]{\frac{P(t)}{Q(t)}}, \sqrt[4]{\frac{Q(t)}{P(t)}}\right\} \frac{E(t)}{\sqrt[4]{P(t_0)Q(t_0)}}r_H(t), \quad t \geq t_0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\tilde{D}m(t)}{\sqrt[4]{a_{12}(t_0)a_{21}(t_0)}} \exp\left\{-\left|\int_{t_0}^t \frac{Re a_{11}(\tau) - Re a_{22}(\tau)}{2}d\tau\right|\right\} \frac{E(t)}{r_H(t)} &\leq u_0(t)+v_0(t) \leq \\ &\leq \frac{\tilde{D}M(t)}{\sqrt[4]{a_{12}(t_0)a_{21}(t_0)}} \exp\left\{\left|\int_{t_0}^t \frac{Re a_{11}(\tau) - Re a_{22}(\tau)}{2}d\tau\right|\right\} E(t)r_H(t), \quad t \geq t_0. \end{aligned}$$

Taking into account the equalities $(u(t), v(t)) = \lambda_0(u_0(t), v_0(t))$, $\lambda_0 = const \neq 0$, $|\phi(t_0)| = |\lambda_0|u_0(t_0)$, $|\psi(t_0)| = |\lambda_0|v_0(t_0)$ from here we will get

$$D(\phi, \psi) \exp\left\{-\left|\int_{t_0}^t \frac{Re a_{11}(\tau) - Re a_{22}(\tau)}{2}d\tau\right|\right\} \frac{m(t)E(t)}{r_H(t)} \leq |u(t)| + |v(t)|$$

$$\leq D(\phi, \psi) \exp \left\{ \left| \int_{t_0}^t \frac{Re a_{11}(\tau) - Re a_{22}(\tau)}{2} d\tau \right| \right\} M(t) E(t) r_H(t), \quad t \geq t_0. \quad (3.12)$$

From here and from (3.10) it follows (3.7). The assertion i) is proved. Let us prove ii). Let $(\phi(t), \psi(t))$ be a main solution of the system (1.1). Then by (3.1) we have

$$\phi(t) = \phi(t_0) \exp \left\{ \int_{t_0}^t a_{11}(\tau) d\tau \right\} u_0(t), \quad \psi(t) = \psi(t_0) \exp \left\{ \int_{t_0}^t a_{22}(\tau) d\tau \right\} v_0(t), \quad (3.13)$$

$t \geq t_0$, where $(u_0(t), v_0(t))$ is the canonical main solution of the system (1.2). By (2.54) from C) it follows $|\phi_0(t_0)|u_*(t) + |\phi_0(t_0)||v_*(t)| \leq |\phi_0(t_0)|u_*(t_0) + |\phi_0(t_0)||v_*(t_0)|$, $t \geq t_0$. By (3.10) from here and from (3.13) it follows (3.8). The assertion ii) is proved. Let us prove iii). Let $(\phi(t), \psi(t))$ be an ordinary solution of the system (1.1). By (3.1) we have

$$\phi(t) = \exp \left\{ \int_{t_0}^t a_{11}(\tau) d\tau \right\} \left[\lambda_0 u_0(t) + \lambda_* u_*(t) \right], \quad t \geq t_0, \quad (3.14)$$

$$\psi(t) = \exp \left\{ \int_{t_0}^t a_{22}(\tau) d\tau \right\} \left[\lambda_0 v_0(t) + \lambda_* v_*(t) \right], \quad t \geq t_0, \quad (3.15)$$

where $(u_*(t), v_*(t))$ and $(u_0(t), v_0(t))$ are the canonical main and canonical nonprincipal solutions of the system (1.2) respectively, and $\lambda_0 \neq 0$. Then by (2.29) and (2.52) we can deduce from C) that $\tilde{c}_1[u_0(t) + v_0(t)] \leq |\lambda_0 u_0(t) + \lambda_* u_*(t)| + |\lambda_0 v_0(t) + \lambda_* v_*(t)| \leq \tilde{c}_2[u_0(t) + v_0(t)]$, $t \geq t_0$, $\tilde{c}_j = const$, $j = 1, 2$. By virtue of (3.10) from here, from (3.14) and (3.15) we obtain

$$\begin{aligned} \tilde{c}_1 \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 Re a_{jj}(\tau) \right] d\tau - \left| \int_{t_0}^t \frac{Re a_{11}(\tau) - Re a_{22}(\tau)}{2} d\tau \right| \right\} [u_0(t) + v_0(t)] &\leq \\ &\leq |\phi(t)| + |\psi(t)| \leq \\ &\leq \tilde{c}_2 \exp \left\{ \int_{t_0}^t \left[\frac{1}{2} \sum_{j=1}^2 Re a_{jj}(\tau) \right] d\tau + \left| \int_{t_0}^t \frac{Re a_{11}(\tau) - Re a_{22}(\tau)}{2} d\tau \right| \right\} [u_0(t) + v_0(t)], \end{aligned}$$

$t \geq t_0$. By (3.12) from here it follows (3.9). The assertion iii), and therefore, the theorem are proved.

Remark 3.2. Let $a(t)$ and $b(t)$ be the same as in Remark 3.1. Consider the system

$$\begin{cases} \phi'(t) = a(t)\phi(t) + b(t)\psi(t); \\ \psi'(t) = b(t)\phi(t) + a(t)\psi(t), \quad t \geq t_0. \end{cases}$$

For this system we have $\mathcal{F}(t) \equiv 0$, $m(t) = M(t) \equiv 1$. Therefore by Theorem 3.2 for its each nonprincipal solution $(\phi(t), \psi(t))$ the inequalities

$$\begin{aligned} |\phi(t)| + |\psi(t)| &= (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^t [Re a(\tau) + b(\tau)] d\tau\right\} \leq |\phi(t)| + |\psi(t)| \leq \\ &\leq (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^t [Re a(\tau) + b(\tau)] d\tau\right\}, \quad t \geq t_0, \end{aligned}$$

are fulfilled. Hence

$$|\phi(t)| + |\psi(t)| = (|\phi(t_0)| + |\psi(t_0)|) \exp\left\{\int_{t_0}^t [Re a(\tau) + b(\tau)] d\tau\right\}, \quad t \geq t_0,$$

and in this sense the estimates (3.7) are sharp.

Example 3.2. Let us consider the system

$$\begin{cases} \phi'(t) = (-\lambda + \sin t)\phi(t) + t^\alpha\psi(t); \\ \psi'(t) = t^\beta\phi(t) + (-\mu + \cos t)\psi(t), \end{cases} \quad (3.16)$$

$t \geq \frac{\pi}{4}$, where λ , μ , α and β are some real constants. For this system the functions $m(t)$, $M(t)$ and $\mathcal{F}(t)$ are the same, which are in the example 3.1. Applying Theorem 3.2 to (3.16) it is easy to find the following regions of parameters λ , μ , α , β for which Eq. (3.16) is asymptotically stable:

$$\begin{aligned} O_1^0 &\equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + \sqrt{2}, \lambda > 0, \mu > 0, \lambda \neq \mu, \alpha + \beta < 0\}; \\ O_2^0 &\equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 2 + \sqrt{2}, \lambda > 0, \mu > 0, \lambda \neq \mu, \alpha + \beta = 0\}; \\ O_3^0 &\equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| > 3\sqrt{2}, \lambda > 0, \mu > 0, \lambda \neq \mu, \alpha + \beta < 0\}; \\ O_4^0 &\equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 2 > 3\sqrt{2} + 2, \lambda > 0, \mu > 0, \lambda \neq \mu, \alpha + \beta = 0\}; \\ O_5^0 &\equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| = 3\sqrt{2}, \lambda > 0, \mu > 0, \lambda \neq \mu, \alpha = \beta < 0\}; \\ O_6^0 &\equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu > 3|\lambda - \mu| + 1 = 3\sqrt{2} + 2, \lambda > 0, \mu > 0, \lambda \neq \mu, \alpha = \beta = 0\}; \end{aligned}$$

and the following regions of parameters λ , μ , α , β for which eq. (3.16) is instable:

$$O_7^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda \neq \mu, \alpha + \beta > 0\};$$

$$O_8^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| + \sqrt{2} < 2, \lambda < 0, \mu < 0, \alpha + \beta = 0\};$$

$$O_9^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda = \mu < -\frac{\sqrt{2}}{2\pi}\};$$

$$O_{10}^0 \equiv \{(\lambda, \mu, \alpha, \beta) : \lambda + \mu + 3|\lambda - \mu| < 2, |\lambda - \mu| \geq \sqrt{2}, \alpha + \beta = 0\}.$$

As in the case of the system (3.5) the application of the estimates of Liapunov, Yu. S. Bogdanov and estimate by freezing method to the system (3.16) give no result and the estimates by logarithmic norms γ_I and γ_{II} of S. M. Lozinski give result only for $\lambda > 0, \mu > 0, \alpha < -1, \beta < -1$. For the case $\alpha > 0$ or $\beta > 0$ it is impossible by use of the theorem of Wazewski to verify neither asymptotic stability nor instability of system (3,16).

REFERENCES

1. L. M. Liapunov, The general problem of motion stability. Gosudarstvennoe izdatelstvo texnico-teoreticheskoy literatury, Moscow, Leningrad 1950.
2. L. Cezary, Asymptotic behavior and stability of solutions of ordinary differential equations. Moscow, mir, 1964.
3. R. Bellman, Stability theory of differential equations. Moscow, Foreign Literature Publishers, 1954.
4. L. Y. Adrianova, Introduction to the theory of linear systems of differential equations. S. Peterburg, Publishers of St. Petersburg University, 1992.
5. B. P. Demidovich, Lectures on the mathematical theory of stability. Moscow, Nauka, 1967.
6. V. A. Yakubovich, V. M. Starzhinsky, Linear differential equations with periodic coefficients and their applications. Moscow, Nauka, 1972.
7. N. V. Mack - Lachlan, Theory and Applications of Mathieu functions. Moscow, Mir, 1953.
8. D. W. Jordan, P. Smith, Nonlinear differential equations. Oxford University press, 2007.
9. Erbe L. H. Stability results for Periodic second Order Linear Differential Equations. Proc. Amer. Math. Soc., Vol. 93, Num. 2, 1985. pp. 272 - 276.
10. Knovles I. On stability Conditions for Second Order Linear Differential Equations, Journal of Differential Equations 34, 179 - 203 (1979).
11. A. I. Egorov, Uravneniya Rikkati (Riccati equations), Moscow, 2011.
12. G. A. Grigorian, Two comparison criteria for the scalar Riccati equations and some their applications. Izv. vuzov, Matematika, N° 11, 2012, pp. 20 - 35.
13. G. A. Grigorian, On two comparison tests for second-order linear ordinary differential equations (Russian) Dier. Uravn. 47 (2011), no. 9, 1225 - 1240; translation in Dier. Equ. 47 (2011), no. 9 1237 - 1252, 34C10.
14. G. A. Grigorian, On some properties of solutions of the Riccati equation. Izvestiya NAS of Armenia, vol. 42, N° 4, 2007, pp. 11 - 26.