
IMAGE: A MAP OF THE STARS OF THE ORION CONSTELLATION

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# Theory of Dark Energy

*Friedhelm M. Jöge*

## ABSTRACT

The presented article provides a theory of dark energy that appears to have been developed in two complementary ways. On the one hand, this theory is based on physics and mathematics and, on the other hand, it is developed on the basis of available data. This corresponds to the discovery of the laws of planetary motion in elliptical planetary orbits by JOHANNES KEPLER in the past. He developed his laws from a large amount of data. Later it was theoretically substantiated more thoroughly by ISAAK NEWTON. The focus is on deriving a formula for the equivalence of energy and time (1), page 2 . Precursors to the presented „Theory of Dark Energy“ were published in the articles [1-11].

The derivation of the formula for the equivalence of energy and time provides new theoretical insights and applications in theoretical terms. These are listed in the “Application“ and „Future research fields“ sectors; five applications are listed in the „Application“ sector.

This derivation leads to the discovery of a new law of nature. This is explained in section „Conclusion“.

A formula for calculating dark energy was developed in a previous article published in the International Journal of Physics and Astronomy [1].

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Friedhelm M. Jöge*


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*A formula for calculating dark energy was developed in a previous article published in the International Journal of Physics and Astronomy [1].*
*It is:*

$$E_d = (h/t_p^2) \cdot t_u \qquad (1.2)$$

*The „The Foundations of a Dark Energy Theory" was first mentioned in my previous work "Commentary about Calculation of Dark Energy and Dark Matter", published in the Journal of Physics and Astronomy [2]. The presented article „Theory of Dark Energy" is now completed and a physical-mathematical and theoretical derivation of the formula (1.2) is provided.*


*Keywords:* dark energy, planck time, law of nature, age of the universe, fundamental oscillations of a cosmic space, cosmology, theoretical physics.

*First way*

## I. PHYSICAL-MATHEMATICAL AND THEORETICAL DERIVATION

The derivation of the formula (1), page 2 for the „Equivalence of Energy and Time" [3] requires only the assumptions that the PLANCK time $t_p$ is an oscillation period $\tau$ and dark energy satisfies the PLANCK / EINSTEIN formula

$$E = h\,\nu \qquad (1.1)$$

Oscillations are fundamental oscillations of a cosmic space [4 pg.15]. THOMAS GÖRNITZ says: „Structural quanta emerge from a quantum-theoretical description of „oscillation states" of a system

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around its ground state. They produce many effects. The AQIs of protyposis are also structural quanta and not particles. One can interpret them as the „fundamental oscillations of the cosmic space".

---

for the equivalence of energy and time then this leads to:

With $\nu = 1/\tau$ , you get

$$E = h/\tau$$

With $\tau = t_p$ , you get

$${}_pE = h/t_p \quad \text{for Energy in PLANCK time}$$

$${}_1E = (h/t_p^2) \quad \text{for Energy in 1 s}$$

$$E = (h/t_p^2) \cdot t \qquad (1) \quad \textit{Equivalence of Energy and Time}$$

For the age of the universe $t_u$ , you get

$$E_d = (h/t_p^2) \cdot t_u \qquad (2) \quad \textit{Equivalence of dark Energy and age of the universe}$$

*Second way*

## II. DERIVATION WITH DATA

In my article „Calculation of Dark Energy and Dark Matter" [1] you can find on page 2 the derived formula:

$$E_M = c^5 / (8^{1/2}\, G\, H_o) = 5.61 \cdot 10^{69}\ J \qquad (2.1)$$

This formula did emerge from the BECKENSTEIN HAWKING entropy and the HAWKING temperature, see my article [1], pg.2.

$$\text{In formula (1.2):} \quad (h/t_p^2) \text{ is } = 2.2802 \cdot 10^{53}\ Js^{-1} \qquad (a)$$

$$E_d = 5.61 \cdot 10^{69}\ J \cdot 70 / 4 = 0.982 \cdot 10^{71}\ J$$

With $t_u = 4.3056 \cdot 10^{17}$ s , you get

$$E_d/t_u = 0.982 \cdot 10^{71}\ J / 4.3056 \cdot 10^{17}\ s = 2.2807 \cdot 10^{53}\ Js^{-1} \qquad (b)$$

The numerical values calculated using formulas (a) and (b) correspond to a high degree.

This means that formula (1.2) is validated and correct. It should be acknowledged as a law of nature, so as KEPLER's laws of planetary orbit descriptions have been confirmed and acknowledged as correct from the large amounts of data available.

The available data has been published by the MAX PLANCK Institute for Radio Astronomy.

## III. APPLICATION

Applications of the formula (1.2) as natural law for experimental research or practical applications have not yet been carried out. The reasons for this are explained in „Future research fields" in the next section. However, applications to answer open questions in Theoretical Physics can be made.

The following publications show how formula (1.2) can be used to answer open questions and give concrete examples of such applications.

In addition to the four applications previously described in the article „Time is quantized“ [5], „The Universe – an Open System“ [6], „Dak Energy is not constant“ [7], „Equivalence of Information and Squared Energy“ [8] a the present article “Theory of Dark Energy” also contains an application of formula (1.2).The statement of Prof. Dr. Alexandre Tkatchenko from the University of Luxemburg also contains a possible application of formula (1.2).The application in the present article „Theory of Dark Energy“ should be highlighted.

The possible application in this case consists in that what Prof. Dr. Alexandre Tkatchenko says: „Accurate calculating the value of Dark Energy could be helps bring together two of the largest fields in physics: Quantum Field Theory (QFT) and the General Theory of Relativity (ART) developed by ALBERT EINSTEIN.

*Future research fields*

Future experimental applications of formula (1.2) are hardly to be expected, as dark energy is not yet experimentally accessible. In addition, dark energy cannot be observed directly and is diffusely distributed throughout the universe and is therefore not easy to detected.

However, applications of formula (1.2) could be made to answer open questions in Theoretical Physics: Since the dark energy is relative [9] and dark energy is not constant [7], the energy on Earth is different than the energy at the edge of the universe. What this means for the development of the universe from Big Bang to today must be researched. That doesn't matter for the Earth, but whether the linear function of dark energy depending on the age of the universe (see diagram [7]) is still valid and the exact calculating of dark energy is still correct must be reconsidered.

Another application of the formula (1.2), which was already mentioned in the „Application“ section, is given by Prof. Dr. Alexandre Tkatchenko.

Research into possible interdisciplinary applications of formula (1.2) could, for example, be applied in areas outside of physics, such as in cosmology or in the interdisciplinary modeling of physical systems, in future research.

Expanding the possible scope of application could open up exciting avenues for further research.

## IV. CONCLUSION

The formula (1.2) is theoretically justified and validated based on available data.

It should be acknowledged as a natural law. „KEPLER's“ laws of planetary motion were theoretically founded by ISAAK NEWTON („NEWTON's“ law of gravitation), which he discovered and which represents a law of nature. The situation is similar when generalizing the formula for the „Equivalence of Dark Energy to the age of the universe“ to the „Equivalence of Energy and Time“ [3]. That is, when I say in all modesty: „This formula also represents a law of nature“.

The article „Calculation of Dark Energy and Dark Matter“ [1] was the first to accurately calculate the value of Dark Energy. Accurate calculating of this value could help bring together two of the largest fields in physics: Quantum Field Theory (QFT) and the General Theory of Relativity (ART), developed by ALBERT EINSTEIN. That's what Prof. Dr. Alexandre Tkatchenko says. This is also a possible application.

*Definition of symbols used in formulas*

E = Energy
$E_d$ = dark Energy
$E_M$ = Energy that corresponds to the visible baryonic matter
$t_u$ = age of the universe = 13.75 billion years with 1 year = 365, 2422 days (Google).
$t_u = 4{,}30557 \cdot 10^{17}$ s [12]
$t_p$ = PLANCK time
h = PLANCK constant of action $\hbar = h/(2\pi)$
G = constant of gravitation
Ho = HUBBLE constant
ν = frequency
τ = period of oscillation

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12. This is the result of a new detailed study. Researchers from the University of Bonn evaluated images from the HUBBLE Space Telescope together with colleagues from the US- universities of Stanford and California.


*The highlight:*

your calculation takes more factors into account than previous studies. Their value for the age of the universe is therefore particularly close to reality. The results will soon be published in the trade magazine „Astrophysical Journal“.

# A General Method for Construction of Bivariate Stochastic Processes Given Two Marginal Processes

*Jerzy K. Filus & Lidia Z. Filus*

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## ABSTRACT

Given two arbitrary univariate stochastic processes $\{Y_t\}$, $\{Z_t\}$, assumed to only share the same time t. When considered as describing (time dependent) random quantities that are physically separated (the baseline case), the processes are independent for every time epoch t. From this trivial case we move to the case where physical interactions between the quantities make them, at any moment t, stochastically dependent. For each time epoch t, we impose stochastic dependence on two "initially independent" random variables $Y_t$, $Z_t$ by multiplying the product of their survival functions by a proper 'dependence factor' $\varphi_t(y_t,z_t)$ , obtaining in this way a universal ("canonical") form of any (!) bivariate distribution (in some known cases, however, this form may become complicated thou it always exists). This factor, basically, may have the form $\varphi_t(y,z) = \exp[ - \int_o^y \int_o^z \psi_t(s;u)\, dsdu ]$ whenever such a function $\psi_t(s;u)$ exists, for each t. That representation of stochastic dependence by the functions $\psi_t(s;u)$ leads, in turn, to the phenomenon of change of the original (baseline) hazard rates of the marginals, similar to those analyzed by Cox (1972) and, especially Aalen (1989) for single pairs (or sets) of, time independent, random variables.

*Keywords:* bivariate survival functions, bivariate stochastic processes' constructions, dependence functions, biomedical applications, econometrics, bivariate Wiener and Pareto stochastic processes construction.

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## ABSTRACT

*Given two arbitrary univariate stochastic processes $\{Y_t\}$, $\{Z_t\}$, assumed to only share the same time t. When considered as describing (time dependent) random quantities that are physically separated (the baseline case), the processes are independent for every time epoch t. From this trivial case we move to the case where physical interactions between the quantities make them, at any moment t, stochastically dependent. For each time epoch t, we impose stochastic dependence on two "initially independent" random variables $Y_t$, $Z_t$ by multiplying the product of their survival functions by a proper 'dependence factor' $\varphi_t (y_t,z_t)$ , obtaining in this way a universal ("canonical") form of any (!) bivariate distribution (in some known cases, however, this form may become complicated thou it always exists). This factor, basically, may have the form $\varphi_t (y,z) = exp[ - \int_o^y \int_o^z \psi_t (s;u) dsdu ]$ whenever such a function $\psi_t (s;u)$ exists, for each t. That representation of stochastic dependence by the functions $\psi_t (s;u)$ leads, in turn, to the phenomenon of change of the original (baseline) hazard rates of the marginals, similar to those analyzed by Cox (1972) and, especially Aalen (1989) for single pairs (or sets) of, time independent, random variables. That is why, until Section 4, we would rather consider single random vectors (Y, Z)' joint survival functions, mostly as a preparation to the theory of bivariate stochastic processes $\{(Y_t, Z_t)\}$ constructions as initiated in Section 4.*

*The bivariate constructions are illustrated by examples of some applications in biomedical and econometric areas. Reliability applications, associated with the considered "micro shock □ microdamage" paradigm, obviously may follow.*


*Keywords:* bivariate survival functions, bivariate stochastic processes' constructions, dependence functions, biomedical applications, econometrics, bivariate Wiener and Pareto stochastic processes construction.

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## I. INTRODUCTION

Suppose two random variables Y, Z are given, with some "real life" interpretation.

Mainly we consider a pair Y, Z to have bio-medical, reliability or econometric interpretation. For example, Y, Z may be (stochastically dependent) levels of some chemicals (hormones, for example) in a human or an animal body. In particular, one may consider Y to describe random level of "bad" cholesterol and Z an accompanying sugar level (both measured at the same time). Apparently, the two (random) quantities are stochastically dependent on each other. Another tandem of random quantities Y, Z can be daily salt consumption Y and blood pressure Z. There are a countless number of such pairs of random quantities that have biomedical meanings. They mostly are stochastically dependent. This means that a measured value z' of Z has an influence on the probabilities $P(Y \geq y_o)$ (if, for a fixed value z, the random event $Z \geq z$ occurs and z', such that $z' \geq z$, is its elementary realization). In particular, $y_o$ may be considered a 'critical value' of Y (for example, the critical cholesterol level $y_o$) so that occurrence of the random event $Y \geq y_o$ means occurrence of a disease.

Again, the probability of occurring of the so defined disease may depend on the measured value z' of the random quantity Z when, for a given measurement z', we observe that $z' \geq z$, for some fixed value z. Here notice that, in this framework, one may disregard an obtained by a measurement particular value z' and only notice if $z' \geq z$ or otherwise. The more essential value, in this case, is z.

In the general case of two random variables Y, Z we will be interested in finding analytical formulas for all the conditional probabilities $P(Y \geq y \mid Z \geq z)$ for any arbitrary value y of Y and not only for a critical one $y_o$ . To achieve this goal we will outline methodology for construction of the joint probability distribution of the random variables Y, Z under various situations.

A similar but different development of methodology of construction of bivariate probability distributions based on Cox and Aalen ideas (see, [2] and [1] ) can be found in [3].

The constructions so far rely on building bivariate probability distributions of random vectors. However, the main goal of this work is to extend the obtained methods to methods where the random variables Y, Z are replaced by t-dependent stochastic processes $\{Y_t\}$, $\{Z_t\}$. Thus, given two marginal (originally independent) univariate stochastic processes, we provide tools for constructing a class of bivariate processes $\{(Y_t, Z_t)\}$ such that the "initial" processes $\{Y_t\}$, $\{Z_t\}$ and their probability distributions, for each t, remain as the marginals within the constructed joint. The obtained methods of the constructions turn out to be quite general and are governed by some (linear, in particular) integral equations.

As an example of the construction of processes with continuous time we consider construction of some classes of "bivariate Wiener processes" based on two given Wiener univariates. In applications, the underlying univariate Wiener processes may have econometric meanings. For example, one of the two marginal processes may describe the stock market level time evolution while the other the accompanying gross national product rate at time t. In another application, a process describes the level of inflation at a given time and the other processes the level of employment at the same time. In all cases one is interested in the stochastic dependence between the two marginal processes. That need motivates constructions of the 'joint stochastic processes'. In general, the marginal processes, as applied in econometry, are normal. However, among all the mutual stochastic dependencies at any given time epoch t we choose not to be those described by the classical bivariate normal models, since such dependencies are very well known. Our method provides a large variety of stochastic dependencies, some possibly overlooked in literature.

In another example we consider discrete time bivariate processes associated with first Pareto distributions. All the bivariate stochastic processes considerations are included in section 4.
Random Vectors, A General Approach

*2.1* Suppose we are given two arbitrary random variables Y, Z with known marginal survival functions $S_1(y) = P(Y \geq y)$ and $S_2(z) = P(Z \geq z)$ . Our aim is to provide a general methodology for constructions of various joint survival functions $S(y,z) = P(Y \geq y, Z \geq z)$ such that all their marginal survival functions are invariant with respect to the performed constructions, always remaining the same as the originally given $S_1(y)$ and $S_2(z)$ . In general, we restrict our considerations to the cases where both the probability distributions, given by the survival functions $S_1(y)$ , $S_2(z)$ , possess probability densities so that the corresponding hazard rates $\lambda_1(y)$ , $\lambda_2(z)$ exist too. In this case, according to the common rule, we can express the considered marginal survival functions as follows:

$$S_1(y) = \exp [ - \int_0^y \lambda_1(t)\, dt ] ,\ S_2(z) = \exp [ - \int_0^z \lambda_2(u)\, du ] . \quad (1)$$

In the simplest case, i.e., when the random variables Y, Z are independent, their joint survival function $S^*(y, z)$ satisfies:

$$S^*(y, z) = S_1(y)\, S_2(z) = \exp [ - \int_0^y \lambda_1(t)\, dt - \int_0^z \lambda_2(u)\, du ] . \quad (2)$$

To impose a dependence structure for the above "initially" independent random variables Y, Z , we multiply the right hand side of formula (2) by a "dependence factor" $\varphi(y,z)$ . This factor we propose to call the "Aalen factor".

2.2 At this point realize that for any joint survival function $S(y,z)$ having the same fixed marginals $S_1(y)$ , $S_2(z)$ there exist unique Aalen factor $\varphi(y, z) = S(y,z) / S_1(y) S_2(z)$ . However, in most situations we may encounter, we do not know in advance the joint survival function $S(y,z)$ but rather we have the marginals $S_1(y)$ , $S_2(z)$ . We aim to choose a proper Aalen factor $\varphi(y, z)$ in order to construct the corresponding $S(y,z)$ . To give some more intuitive meaning to the Aalen factor we express it in exponential form:

$$\varphi(y,z) = \exp [ - \alpha(y,z) ] = \exp[ - \int_0^y \int_0^z \psi(t;u)\, dudt ] .$$

The function $\alpha(y,z)$ we call the "joiner" of the two marginal survival functions $S_1(y)$, $S_2(z)$.

Of course, the joiner $\alpha(y,z)$ is uniquely determined by a given Aalen factor $\varphi(y,z)$ since we have $\alpha(y,z) = - \log \varphi(y,z)$ . However, the joiner $\alpha(y,z)$ not always is represented by the integral $\int_0^y \int_0^z \psi(t;u)\, dudt$ . The 'integral representation' of a joiner is only valid when such a function $\psi(t; u)$ , considered as a *kind of* an "additional hazard rate", exists. Existence of $\psi(t; u)$ together with the existence of the hazard rates $\lambda_1(t)$ , $\lambda_2(u)$ involves existence of the joint probability density of the considered random vector (Y,Z) . Using the function $\psi(t; u)$ we define the conditional hazard rate of $Y \mid Z = u$ to be equal to:
$\lambda_1(t \mid Z = u) = \lambda_1 (t) + \psi(t; u)du$ ,

and the conditional hazard rate of $Z \mid Y = t$ as:

$$\lambda_2(u \mid Y = t) = \lambda_2(u) + \psi(t; u)dt .$$

For the meaning of the differentials $\psi(t;u)du$ and $\psi(t;u)dt$ see section 2.4 and the associated "(microshok ☐ microdamage) ☐ microchange in hazard rate" paradigm. (These infinitesimal "microchanges" correspond to the above considered differentials.)

Since we are mostly dealing with the survival functions (not with probability densities), we will rather be interested in the conditional hazard rates of $Y \mid Z \geq z$ which involve the form:

$$\lambda_1(Y = t \mid Z \geq z) = \lambda_1 (t) + \int_0^z \psi(t;u)\, du , \quad (3)$$

and in the conditional hazard rates of $Z \mid Y \geq y$ using the form

$$\lambda_2(Z = u \mid Y \geq y) = \lambda_2 (u) + \int_0^y \psi(t;u)\, dt . \quad (3^*)$$

The non-negativity assumption for the function $\psi(t;u)$ is not mandatory but simplifies many considerations. More general and also necessary assumption for the class of functions $\psi(t; u)$ are given by the inequalities:

$$\lambda_1(t) + \int_0^z \psi(t;u)\, du \geq 0 \text{ , and}$$

$$\lambda_2(u) + \int_0^y \psi(t;u)dt \geq 0 ,$$

for any nonnegative t, u . As we have already defined it, the left hand sides of the above inequalities represent the (conditional) hazard rates, so they must be non-negative.

Also recall that $\lambda_1(t)$ , $\lambda_2(u)$ are always nonnegative.

Restricting the function $\psi(t;\ u)$ to nonnegative values implies considering positive stochastic dependences only between the considered random variables Y, Z .

2.3 Assuming existence of the underlying hazard rates, let us analyze the general form of the joint survival function under the investigation:

$$S(y,z) = P(Y \geq y\ , Z \geq z) = \exp[\ -\int_0^y \lambda_1(t)\ dt\ \ - \int_0^y \int_0^z \psi(t;u)\ dudt - \int_0^z \lambda_2(u)\ du\ ] \tag{4}$$

with $\psi(t;\ u) \geq 0$.

First realize that upon setting on the right hand side of (4) $z = 0$ , the expression reduces to the marginal $\exp[\ -\int_0^y \lambda_1(t)\ dt\ ] = S_1(y)$ . Likewise, upon the setting $y = 0$ in (4) turns the expression to the marginal $\exp\ [-\ \int_0^z \lambda_2(u)\ du\ ] = S_2(z)$ .

Second, upon dividing both sides of (4) by $S_2(z)$ one obtains the conditional survival function

$$S_1(y \mid z) = P(\ Y\geq y \mid Z \geq z\ ) = \exp\ [\ -\int_0^y \lambda_1(t)\ dt\ \ - \int_0^y \int_0^z \psi(t;u)\ dudt\ ]\ . \tag{5}$$

Also dividing (4) by $S_1(y)$ yields the conditional survival function

$$S_2(z \mid y) = P(Z \geq z \mid Y\geq y\ ) = \exp\ [\ -\int_0^z \lambda_2(u)\ du\ \ - \int_0^y \int_0^z \psi(t;u)\ dudt\ ]\ . \tag{6}$$

2.4 Let us explain more the structure of the conditional hazard rate $\lambda_1(y \mid Z \geq z)$ , given “initially” the marginal (baseline) hazard rate $\lambda_1(y)$. The case of $\lambda_2(z \mid Y \geq y)$ can be understood in an analogous way.
Example 1. Suppose we have two different objects O1, O2 whose “behaviors” are exhibited by the random quantities Y, Z respectively. Take, at first, object O1 as the object “of main interest”, characterized by the quantity Y. Now consider the “activity” of object O2 , measured by the random quantity Z . The activity Z is regarded as “stress” that the object O1 is subjected to.

(Such an ”activity” may be understood as temperature, for example.) In short, the stress Z is an explanatory variable for the variable of the main interest Y. In the *physical* absence of the object O2 the variable Y is independent on Z, so the corresponding hazard rate for Y is simply equal to $\lambda_1(t)$ , and the (unconditioned) survival function of Y equals

$$S_1(y) = P(Y\geq y) = \exp\ [\ -\int_0^y \lambda_1(t)\ dt\ ]\ .$$

( The following ‘micro-shocks □ micro-damages’ mechanism that yields the stochastic dependence is described in more detail in [4].)

When we consider the case where the object O2 is *physically* accompanying (“connected to”) object O1, its activity “produces” at each time epoch ‘*physical*’ micro-shock on O1 . These micro-shocks result in corresponding micro-damages in the physical structure of O1 which in turn result in micro-changes in the hazard rate of the corresponding random quantity Y. Every such a micro-change (as related to a quantity u) at a given moment t relies on adding to the Y’s hazard rate the infinitesimal quantity $\psi(t;\ u)du$ . This change is, actually, too small to be recognized by any physical measurement, but mathematically we can express it as an infinitesimal differential $\psi(t;u)du$ as proportional to the given “intensity” $\psi(t;\ u)$ . The microchanges cumulate as u (for Z = u) “runs” through the interval [0,z] .

This phenomenon of the cumulation of the micro-changes (as corresponding to an 'elementary' random events $Y = t$) results in the conditional hazard rate

$$\lambda_1(t \mid Z \geq z) = \lambda_1(t) + \int_0^z \psi(t;u)\, du \qquad (7)$$

which, evidently, is different from the "original" (baseline) $\lambda_1(t)$.
The foregoing integration relates to the transfer from an elementary event $Z = u$ to the random event $0 \leq Z < z$ [ Here realize that since we apply, as the analytic tool, the survival functions, instead of the distribution functions, we eventually will '*calculate*' the probability of its complement, i.e., the probability of $Z \geq z$ ].

Integrating (7) with respect to "time" t over the interval [0,y] and applying formula (5), we obtain the conditional survival function $P(Y \geq y \mid Z \geq z)$. The latter, once multiplied by the marginal distribution in the form $P(Z \geq z)$, results again in the joint survival function of the random vector (Y,Z).
Setting $\int_0^z \psi(t;u)\, du = \varphi(t,z)$ we obtain the formula

$$\lambda_1(t \mid Z \geq z) = \lambda_1(t) + \varphi(t,z), \qquad (7^*)$$

which corresponds to the Aalen additive model [1] being the modification of the famous Cox proportional hazards model [2].

*Remark 1:* In a more general case one may consider the second differential $\psi(t;u)dtdu$ of the joiner α ( , ) as a microchange (corresponding to both elementary events ["actions"] t and u [or u at the time epoch t] ) in both hazard rates caused by mutual interaction of the "micro-shocks" described by t and u, caused by activities of the objects O1 and O2. Realize that neither t nor u needs to represent time (they may, for example, be levels of hormones in a human body).

In this framework, that interaction may be better understood if the intensity $\psi(t; u)$ is chosen to have separated variables, i.e., $\psi(t; u) = \psi_1(t)\, \psi_2(u)$ (see examples in below). Notice also that the secondary differential $\psi(t;u)\, dtdu$ is added to both the original hazard rates $\lambda_1(t)$ and $\lambda_2(u)$, which agrees with the idea of *mutual* interactions.

On Representation of Bivariate Survival Functions

### 3.1 The Representation

Write formula (4) in a slightly more general form:

$$S(y,z) = P(Y \geq y, Z \geq z) = \exp[ -\int_0^y \lambda_1(t)\, dt - \alpha(y,z) - \int_0^z \lambda_2(u)\, du . \qquad (4^*)$$

We now depart from the Aalen-like model towards a more general paradigm. We even may drop the assumption on the existence of the hazard rates $\lambda_1(y)$, $\lambda_2(z)$, and the function $\alpha(y,z)$ need not to be expressed by the integral $\int_0^y \int_0^z \psi(t;u)\, dudt$. Instead of formulas (4) and (4*) for the joint survival function of (Y,Z) we may use the following, more general, formula:

$$S(y, z) = S_1(y)\, S_2(z) \exp[ -a(y,z) ], \qquad (8)$$

where $\alpha(y,z)$ is assumed to be a real function, defined for $y \geq 0$, $z \geq 0$.

We will require that it satisfies the conditions specified as follows. Thus the function $\alpha(y,z)$ is:

(1) continuous, for each z, with respect to y and for each y with respect to z,
(2) nondecreasing in y and z,
(3) $\alpha(0,z) = \alpha(y,0) = 0$.

From the last condition we directly obtain the following Property.

*Property 1:* If the latter condition (3) is satisfied, then both the marginal probability distributions of the joint distribution given by formula (8) are preserved in the sense that they are the same as the baseline distributions $S_1(y)$ , $S_2(z)$ originally given.

If the marginal and the joint probability densities of the considered random variables Y, Z exist, the concern is on non-negativity of them. While the marginal densities, say, f(y) , g(z) are non-negative due to the common assumption, we need some special condition for $\alpha(y,z)$ to assure non-negativity of the corresponding joint density $h(y,z)$ .

For the joint density $h(y,z)$ to exist, the function $\alpha(y,z)$ must have continuous partial derivatives of first order $\alpha_y(y,z)$ , $\alpha_z(y,z)$ and the continuous second order mixed partial derivative $\alpha_{y,z}(y,z) = \psi(y,z)$ .

An additional condition that must be satisfied by $\alpha(y,z)$ has the form of the following inequality:

$$[\ \lambda_1(y) + \alpha_y(y,z)\ ] \cdot [\ \lambda_2(z) + \alpha_z(y,z)\ ] \geq \alpha_{y,z}(y,z) = \psi(y,z)\ . \qquad (9)$$

This follows from the form of the joint density (if it exists):

$$h(y,z) = \partial^2/\partial y \partial z\ S(y,z)$$

$$= \{\ [\ \lambda_1(y) + \alpha_y(y,z)\ ]^{\,x}\ [\ (\lambda_2(z) + \alpha_z(y,z)\ ] - \alpha_{y,z}(y,z)\ \}\ \exp[-\ \Lambda_1(y) - \Lambda_2(z) - \alpha(y,z)\ ]\ ,$$

which must be nonnegative. Here, $d/dy\ \Lambda_1(y) = \lambda_1(y)$ , and $d/dz\ \Lambda_2(z) = \lambda_2(z)$ , and $\alpha_{y,z}(y,z) = \psi(y,z)$ .
A simpler condition than (9) for the non-negativity of h(y,z) , which is sufficient and necessary too, for the existence of the joint survival function (4*) is the following, obtained
from (9):

$$\lambda_1(y)\ \lambda_2(z) \geq \alpha_{y,z}(y,z) = \psi(y,z)\ . \qquad (9^*)$$

The condition (9) or (9*) together with the conditions (1), (2), and (3) are sufficient and necessary for "connecting" the two survival functions $S_1(y)$ , $S_2(z)$ by a given joiner $\alpha(y,z)$ into the bivariate model S(y, z) .

As it was pointed out above, there may be a numerous of such models.

### *3.2 Particular Cases of the Bivariate Models*

A. Obviously, when $\alpha(y,z)$ reduces to zero for all y , z , then model (8) describes independent random variables.
B. If the baseline hazard rates exist and are constant, we call this model "exponential".
In this case we may choose $\psi(y,z) = a$ = constant to obtain the following special case
of model (4*):

$$S(y,z) = \exp\ [\ -\ \lambda_1\ y\ -\ ayz - \lambda_2\ z\ ]\ , \qquad (10)$$

where $0 \leq a \leq \lambda_1 \lambda_2$ .

This, apparently, represents the first bivariate exponential Gumbel probability distribution (see [5]).

C. One also obtains the following 'Weibullian version' of the above bivariate Gumbel:

$$S(y,z) = \exp[-\lambda_1 y^{\gamma_1} - a y^{\gamma_1} z^{\gamma_2} - \lambda_2 z^{\gamma_2}], \quad (11)$$

where $\gamma_1$ and $\gamma_2$ are positive reals.

The latter two models are special cases of representation (4*).
We only need to check when condition (9*) is satisfied, i.e.,

$$\lambda_1(y)\,\lambda_2(z) \geq \psi(y,z). \quad (11^*)$$

To check for that, we properly differentiate the terms of the expression $-\lambda_1 y^{\gamma_1} - a y^{\gamma_1} z^{\gamma_2} - \lambda_2 z^{\gamma_2}$ ( $\lambda_1 y^{\gamma_1}$ over y and $\lambda_2 z^{\gamma_2}$ over z and $a y^{\gamma_1} z^{\gamma_2}$ over y and z) , and set inequality (11*) in the form:

$$\lambda_1\lambda_2\,\gamma_1\,\gamma_2\;\; y^{\gamma_1-1} z^{\gamma_2-1} \geq a\,\gamma_1\,\gamma_2\;\; y^{\gamma_1-1} z^{\gamma_2-1}. \quad (12)$$

It holds, for every nonnegative y and z , whenever

$$a \leq \lambda_1\lambda_2.$$

Thus, if the latter condition is satisfied then model (11) is properly defined.

*Remark 2:* As for the above given Weibullian version (11) of the Gumbel exponential model (with $a \leq \lambda_1\lambda_2$) , this can be 'partially' generalized by the following model:

$$S(y,z) = \exp[-\lambda_1 y^{\gamma_1} - a y^{\delta_1} z^{\delta_2} - \lambda_2 z^{\gamma_2}]. \quad (13)$$

Now inequality (12) takes the form:

$$\lambda_1\lambda_2\,\gamma_1\,\gamma_2\;\; y^{\gamma_1-1} z^{\gamma_2-1} \geq a\,\delta_1\,\delta_2\; y^{\delta_1-1} z^{\delta_2-1}. \quad (14)$$

The necessary condition for (14) to hold for all nonnegative values y , z is that

$$1 \leq \delta_i \leq \gamma_i, \text{ for } i = 1, 2, \quad (15)$$

together with

$$a \leq \lambda_1\lambda_2. \quad (16)$$

However, condition (15) makes inequality (14) not true for $0 \leq y, z < 1$ . The remedy for this is to shift the variables y , z by imposing the additional conditions:
$1 \leq c \leq y$ , $1 \leq c \leq z$ for some real c .

Now model (13) is well defined with $1 \leq \delta_i \leq \gamma_i$ . •

*Remark 3:* The form of model (8) and Property 1 allows us to construct numerous of bivariate probability distributions. Namely, realize that:

1. For any given fixed 'joiner function' $\alpha(y,z)$ , 'any' pair of two probability distributions [not necessarily both from the same class of the distributions], determined by $S_1(y)$ , $S_2(z)$ (such that inequality (9) or (9*) is satisfied), may be "connected" by formula (8) to "become" the bivariate survival (distribution) function in which they remain the marginals.

2. For any fixed pair of probability distributions $S_1(y)$ , $S_2(z)$ , there is a wide class of possible joiners $\alpha(y,z)$ [determined by the conditions (1), (2), (3) from section 3.1 together with inequality (9) or (9*)] such that the two distributions can be "connected" into the bivariate model (8) in as many ways as there are possible proper functions $\alpha(y, z)$ .

Finally notice that the above methods for "connecting" *any* two probability distributions into bivariate distributions, resembles the idea of the copula methodology [6]. It is, however, different. For more remarks on that see [3]. •

## II. ON CONSTRUCTION OF BIVARIATE STOCHASTIC PROCESSES GIVEN ANY TWO UNIVARIATE MARGINAL PROCESSES WHICH SHARE THE SAME TIME.

Suppose the (marginal) stochastic processes $\{Y_t\}$, $\{Z_t\}$ are completely defined. So, for every time epoch $t \in T$ (T is a nonempty set) two random variables $Y_t$, $Z_t$ have known in advance survival functions
$S_t(y) = P(Y_t \geq y)$ ,
$R_t(z) = P(Z_t \geq z)$ .

If the corresponding hazard rates $\lambda_1(y, t)$, $\lambda_2(z, t)$ exist, then, according to what was pointed out in previous sections, for any joiner function $\alpha_t(y,z) = \alpha(y,z,t)$ [satisfying, for each t, inequality (9) with respect to the $\lambda_1(y,t)$ , $\lambda_2(z, t)$ as well as conditions (1) – (3) from section 3.1] there exist a unique joint survival function which is also a function of time t, given by:

$$S_t(y,z) = P(Y_t \geq y , Z_t \geq z) = S_t(y)\, R_t(z) \exp[ - \alpha(y,z,t) ].$$

Our intention is to consider, for each $t \in T$, the function $S_t(y, z)$ as the joint survival function (or joint distribution) of the bivariate stochastic process $\{ (Y_t, Z_t) \}_{t \in T}$ . At this point recall that the marginals $S_t(y)$, $R_t(z)$ [as functions of time t] are assumed to define completely the processes $\{Y_t\}$, $\{Z_t\}$ respectively.

Suppose time t is continuous. If both survival functions $S_t(y)$ and $R_t(z)$ are continuous functions of the time then we will postulate the joiner $\alpha(y,z,t)$ to be continuous, as a function of t, as well . Now consider two examples of bivariate stochastic processes whose constructions are based on known initial univariate (marginal) processes.

*Example A:* Consider two Wiener stochastic processes $\{Y_t\}$, $\{Z_t\}$ with the, given in advance, for each t, survival functions

$$P(Y_t \geq y) = 1 - \varphi( (y - bt) / \sigma_1\sqrt{t} ) \quad \text{and} \tag{17}$$

$$P(Z_t \geq z) = 1 - \varphi( (z - ct) / \sigma_2\sqrt{t} ) , \tag{17*}$$

where $\varphi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} \exp[ -w^2 / 2]\, dw$, and b, c are the real parameters of the considered Wiener processes.

For all time epochs s , t such that $0 < s < t$ , all the distributions of the random vectors $(Y_s, Y_t)$ and $(Z_s, Z_t)$ are described by the usual bivariate formulas associated with Wiener processes.

We seek, for every time epoch t, a joiner $a(y,z,t)$ for the two survival functions (17), (17*) which analytically is nice and simple enough, also as a function of t . It needs to satisfy conditions (1) – (3) [section 3.1] as well as inequality (9) or (9*) for each t . If found, then, for each $t > 0$, the joint survival function of the stochastic process $\{ (Y_t, Z_t )\}$ would have the form (8) which, in this case, is:

$$S_t(y,z) = S(y,z,t) = P(Y_t \geq y)\, P(Z_t \geq z) \exp[ - \alpha(y,z,t)] . \tag{18}$$

Of course, in the vast majority of cases, (18) is different from *classical* bivariate normal distributions even if the marginals are normal.

Realize that conditions (1) – (3) from section 3.1 are usually easily checked to be satisfied, so that we really need to check condition (9).

Recall, in the case of stochastic processes, this amounts to checking if:

$$\partial^2 /\partial y\partial z\, \alpha(y,z,t) = \alpha_{yz}(y,z,t) = \Psi(y,z,t)$$
$$\leq [\lambda_1(y,t) + \alpha_y(y,z,t)] \times [\lambda_2(z,t) + \alpha_z(y,z,t)]. \quad (19)$$

This inequality needs to be satisfied for each t.

At this point notice that**:**

$$\alpha_y(y,z,t) = \int_{-\infty}^{z} \Psi(y,v,t)\, dv \text{ and}$$

$$\alpha_z(y,z,t) = \int_{-\infty}^{y} \Psi(u,z,t)\, du .$$

These quantities, upon the additional assumption $\Psi(u,v,t) = 0$ for all nonnegative u, v, are nonnegative and therefore the inequality

$$\lambda_1(y,t)\,\lambda_2(z,t) \leq [\lambda_1(y,t) + \alpha_y(y,z,t)] \times [\lambda_2(z,t) + \alpha_z(y,z,t)]$$

always holds.

The latter together with (19), provides solutions of (19) in the form:

$$\Psi(y,z,t) = \omega(t)\,\lambda_1(y,t)\,\lambda_2(z,t),$$

where the coefficient function $\omega(t)$ satisfy $0 \leq \omega(t) \leq 1$, and, in particular, $\omega(t)$ may be considered a constant in t, especially one may set $\omega(t) = 1$ for each t.

The model (or rather a candidate for a model in a practical application) we have obtained, has the form of the time dependent joint survival function of the random vectors $(Y_t,Z_t)$ for $t \geq 0$. It has the following form:

$$S_t(y,z) = P(Y_t \geq y)\, P(Z_t \geq z) \exp[-\omega(t) \int_{-\infty}^{y} \int_{-\infty}^{z} \lambda_1(u,t)\,\lambda_2(v,t)\, du\, dv]$$
$$= \exp[-\int_{-\infty}^{y} \lambda_1(u\, t)du - \omega(t) \int_{-\infty}^{y} \int_{-\infty}^{z} \lambda_1(u,t)\,\lambda_2(v,t)\, du\, dv - \int_{-\infty}^{z} \lambda_2(v,t)dv].$$

For the above considered '*bivariate Wiener stochastic process*' the marginals $P(Y_t \geq y)$ and $P(Z_t \geq z)$ present in the last formula, are given by (17) and (17*).

Also, in the above case, $\lambda_1(u,t)$, $\lambda_2(v,t)$ are the corresponding hazard rates associated with the considered normal distributions. It is clear, however, that the obtained class of models is much wider than the class of the bivariate Wiener. Write inequality (19) in the form:

$$\Psi(y,z,t) \leq [\lambda_1(y,t) + \int_{-\infty}^{z} \Psi(y,v,t)dv] \times [\lambda_2(z,t) + \int_{-\infty}^{y} \Psi(u,z,t)du]. \quad (19^*)$$

Another candidate for a set of the models (i.e., set of "bivariate Wiener stochastic processes") will be given by the set of the functions $\Psi(y,z,t)$ that are solutions of the following integral equation directly derived from inequality (19*):

$$\Psi(y,z,t) = [\lambda_1(y,t) + \omega_1(t) \int_{-\infty}^{z} \Psi(y,v,t)dv] \cdot [\lambda_2(z,t) + \omega_2(t) \int_{-\infty}^{y} \Psi(u,z,t)du], \quad (20)$$

Where $0 \leq \omega_i(t) \leq 1$, for i = 1, 2 .

Realize that the right-hand side of (20) is always less than or equal than the right hand side of (19*). For equation (20) to hold the condition $\Psi(y,z,t) \geq 0$ is , in general, essential.

The integral equation (20) is nonlinear. However, it can be simplified to the following:

$$\Psi(y,z,t) = g(y,z,t) + \omega_1(t)\,\lambda_2(z,t) \int_{-\infty}^{z} \Psi(y,v,t)dv + \omega_2(t)\,\lambda_1(y,t) \int_{-\infty}^{y} \Psi(u,z\,t)du\,, \qquad (20^*)$$

where $0 \leq g(y,z,t) \leq \lambda_1(y,t)\,\lambda_2(z,t)$ is any continuous function.

Equation (20*) is linear (nonhomogeneous), but the (in general, known) coefficients are still variable.

The set of solutions of (20*) is contained in the set of solutions of equation (19*) , so that any solution of (20*) is a solution of (19*).

Equation (20*) can be simplified more. This will yield a smaller set of solutions being still solutions of (20*). Namely, setting $g(y,z,t) = 0$, we obtain the (purely) linear integral equation

$$\omega_1(t)\,\lambda_2(z,t) \int_{-\infty}^{z} \Psi(y,v,t)\,dv + \omega_2(t)\,\lambda_1(y,t) \int_{-\infty}^{y} \Psi(u,z,t)\,du = \Psi(y,z,t)\,. \qquad (20^{**})$$

So the set of solutions of (20**) forms a vector space over the field of real numbers. In the narrowest version of the problem the coefficients can be made constants. Especially the assumptions $\lambda_1(y,t) = \lambda_1$ and $\lambda_2(z,t) = \lambda_2$ comprise the exponential case. But to stick with the "Wiener model" one must keep the hazard rates (coefficients) $\lambda_1(y,t)$ , $\lambda_2(z,t)$ in (20*) and in (20**) as corresponding to the distributions given by (17) and (17*).

Now, having any solution $\Psi(y,z,t)$ of any integral equation above, we find the joint survival function of the corresponding bivariate stochastic process as expressed by formula (18), where $\alpha(y,z,t\,) = \int_{-\infty}^{y} \int_{-\infty}^{z} \Psi(s,r,t)\,dsdr$ . The so obtained joiner $\alpha(y,z\,t)$ (or $\alpha(y,z)$ in the case of random vectors only) would be a candidate for the bivariate stochastic model.

Next steps are statistical verifications of an eventual fit of the obtained models to a given data that might yield the eventual choice of the best solution among all the obtained. This subject is, however, out of scope of this paper.

Notice that the above presented method for construction of bivariate Wiener processes is applicable to any two (marginal) stochastic processes $\{Y_t\}$ , $\{Z_t\}$ such that each random variable $Y_t$ and $Z_t$ possesses a hazard rate. Thus, one obtains more models. The method also includes discrete time cases like the case described below.

*Example B.* In this example we seek an additional method for constructions. From now on we slightly change the notation by replacing the symbols (Y, Z) for the random vectors by the symbols (X,Y) . Consider the following first Pareto survival functions for two random variables X , Y :

$$S(x) = P(X \geq x) = (x_0 / x)^{\alpha}\,,$$

$$R(y) = P(Y \geq y) = (y_0 / y)^{\beta}\,,$$

$$\text{where} \quad x \geq x_0 > 0\,,\ y \geq y_0 > 0\,,\ \alpha > 0\,,\ \beta > 0\,.$$

In this (Pareto) case the variables X , Y usually (but not always) describe income or wealth redistributed within a society. We consider the case where the same society is subdivided in some manner into two groups that differ by professions or some other social indicator (ethnicity, race, gender, age etc … ). In the models the differences between the two groups income redistributions are reflected by the parameters $x_o$ , $y_o$ (minimal incomes), and by the shape parameters $\alpha$ , $\beta$ . It is reasonable to assume that the probability distributions of the incomes together with the incomes themselves will evolve over time. That is why we consider stochastic (Pareto) processes $\{X_t\}$ , $\{Y_t\}$ description, where the discrete time t is defined as multiplicities $t = r, 2r, 3r, \ldots$ of some time period r such as a year, a quarter, or a month. In our notation we adopt $r = 1$ , and therefore $t = 1, 2, 3 \ldots$ . We now assume that, for each t, the random variables $X_t$ , $Y_t$ are distributed according to the rules:

$$S_t(x) = P(X_t \geq x) = (x_o(t) / x)^{\alpha(t)} ,$$

$$R_t(y) = P(Y_t \geq y) = (y_o(t) / y)^{\beta(t)} .$$

For simplicity we will assume that $x_o(t) = x_o$ = constant, and $y_o(t) = y_o$ = constant.

The above univariate "Pareto stochastic processes" are determined by the constants $x_o$ , $y_o$ , and the way we define the functions $\alpha(t)$ , $\beta(t)$ for t = 1, 2, 3, … . Our (simplest) proposition is to define

$$\alpha(t) = A + (t - 1)r , \quad \beta(t) = B + (t - 1)s ,$$

where (as a particular example) $1 \leq A \leq 2$ , $1 \leq B \leq 2$ , $r = 0.10$ , $s = 0.15$ .

Now, for each t, we find the joint survival functions $U_t(x,y)$ of the random vector $(X_t,Y_t)$ . Recall the general form of the joint survival functions:

$U_t(x,y) = S_t(x)\, R_t(y) \exp[ - \alpha^*(x,y,t) ]$ , (here, $\alpha^* \neq \alpha$ and the meaning of $\alpha^*$ is different than that of $\alpha$ ).

Since we already have both $S_t(x)$ and $R_t(y)$ as given above, we need to find a class of proper joiners $\alpha^*(x,y,t)$ to make, for each t, the joint survival function $U_t(x\ y)$ of $(X_t,Y_t)$ properly defined. Set $\alpha^*(x,y,t) = \alpha^*_t(x,y)$.

*Recall that we have:*

$$\alpha^*_t(x,y) = \int_{xo}^{x} \int_{yo}^{y} \Psi_t(u,v)\, dudv,$$

where, for every t, the function $\alpha_t^*(x,y)$ must satisfy conditions (1) - (3) from section 3.1. Since the domain of this function is reduced to $[x_o,+\infty) \times [y_o,+\infty)$ ,

with positive $x_o$ and $y_o$ , condition (3) (section 3.1) now takes the form:

$$\alpha_t^*(x_o,y) = \alpha_t^*(x,y_o) = 0 , \text{ for each } t .$$

Also, for each t, we have $S_t(x_o) = 1$ and $R_t(y_o) = 1$ .

Therefore, as pointed out in Property 1 (section 3.1), it follows that, for each t, the functions $S_t(x)$ , $R_t(y)$ are the marginal survival functions for the joint $U_t(x,y)$ . Moreover, we restrict ourselves to positive stochastic dependences only, so we have, for every t , $\Psi_t(u,v) \geq 0$ . [As already mentioned, the last assumption is not necessary (and not necessarily true in this case), but makes the *initial* considerations clearer.] In order to determine a proper class of the (not necessarily all) joiners $\alpha^*_t(x,y)$ [which are equivalent to the corresponding functions $\Psi_t(u,v)$], we need to examine (for each t) inequality (9*) which is

$$\Psi_t(u,v) \leq \lambda_1(u,t)\, \lambda_2(v,t) , \tag{21}$$

where, in this 'Pareto case', we have $\lambda_1(u,t) = \alpha(t) / u$ and $\lambda_2(v,t) = \beta(t) / v$ .

Thus inequality (21) takes the form

$$\Psi_t(u,v) \leq \alpha(t)\, \beta(t) / uv \text{ , for every } t .$$

Like in Example A, the first easiest solution to our problem is

$$\Psi_t(u,v) = \omega(t)\, \alpha(t)\, \beta(t) / uv ,$$

where, for every $t$, $0 \leq \omega(t) \leq 1$.

Recall at this point that like the variables $x$, $y$ also the variables $u$, $v$ satisfy $u \geq x_o$, $v \geq y_o$.

The natural assumption in this Pareto case is that both $x_o \geq 1$, $y_o \geq 1$ (i.e., the minimal incomes in both social groups are at least 1 "unit"). Taking the above under the consideration we may choose for $\Psi_t(u,v)$ (the determinant of the stochastic dependence between $X_t$ and $Y_t$) the following:

$$\Psi_t(u,v) = \alpha(t)\, \beta(t) / u^{\gamma_1} v^{\gamma_2} , \quad \text{for } \gamma_1 \geq 1 ,\ \gamma_2 \geq 1 .$$

This $\Psi_t(u,v)$ satisfies (21).

Finally, the full version of the so derived joint survival function of the Pareto random vectors $(X_t,Y_t)$, for every $t$, is:

$$U_t(x,y) = S_t(x)\, R_t(y) \exp[\, - \alpha(t)\, \beta(t) \int_{xo}^{x} \int_{yo}^{y} u^{-\gamma_1} v^{-\gamma_2}\, dudv] ,\ \gamma_1 \geq 1 ,\ \gamma_2 \geq 1 .$$

*Annotation:* Just in recent days, when this paper was already finalized, we learned that similar results as for the bivariate distributions (and not bivariate stochastic processes) were considered by Finkelstein [7] in 2003. The author also considered the "Aalen factor" ( see formulas (4) and (5) in [7] ) but under different names and with no reference to Aalen or Cox. Our impression, however, is that the generality of Finkelstein's approach is somewhat limited. Also, in the references he provides, the results are rather specific and the generality or even universality of this bivariate distributions' representation was probably overlooked. But, anyway, the novum of our results became somehow (but not drastically) limited.

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# Research Progress on the Schrödinger Equation that Can Describe the Earth's Revolution and its Applications

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## ABSTRACT

People believe that the Schrödinger equation cannot be used to describe macroscopic objects like the Earth, and Newtonian mechanics cannot be used to describe microscopic systems. The old concept of the relationship between the existing laws of quantum mechanics and classical mechanics undoubtedly has a serious impact on people's understanding of the natural world, the development of physics theories, and the application of existing physics theories. The continuous development of physics theory requires constant changes to some incorrect old concepts. The Schrödinger equation that can describe planetary motion was successfully obtained by replacing the potential energy in the Hamiltonian operator from electromagnetic interaction potential energy to gravitational interaction potential energy. If the distance between the sun and the earth is approximated as a constant, the energy eigenvalues obtained by solving the Schrödinger equation for the Earth's revolution are completely consistent with the results obtained directly using classical mechanics. The direct significance of establishing and applying such equations is that they can simultaneously use classical mechanics and wave dynamics to describe all objects (no longer limited by the mass of the objects), simplifying the calculation process of quantum mechanics.

*Keywords:* planetary model; schrödinger equation; quantum mechanics; classical mechanics; compatibility; meaning of wave function.

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## ABSTRACT

*People believe that the Schrödinger equation cannot be used to describe macroscopic objects like the Earth, and Newtonian mechanics cannot be used to describe microscopic systems. The old concept of the relationship between the existing laws of quantum mechanics and classical mechanics undoubtedly has a serious impact on people's understanding of the natural world, the development of physics theories, and the application of existing physics theories. The continuous development of physics theory requires constant changes to some incorrect old concepts. The Schrödinger equation that can describe planetary motion was successfully obtained by replacing the potential energy in the Hamiltonian operator from electromagnetic interaction potential energy to gravitational interaction potential energy. If the distance between the sun and the earth is approximated as a constant, the energy eigenvalues obtained by solving the Schrödinger equation for the Earth's revolution are completely consistent with the results obtained directly using classical mechanics. The direct significance of establishing and applying such equations is that they can simultaneously use classical mechanics and wave dynamics to describe all objects (no longer limited by the mass of the objects), simplifying the calculation process of quantum mechanics. It has been proven that classical mechanics and wave dynamics are compatible. It has been proven that classical mechanics and wave dynamics are compatible, and there is no insurmountable gap between them. This result has a huge positive impact on the theoretical updates and applications of quantum mechanics.*


*Keywords:* planetary model; schrödinger equation; quantum mechanics; classical mechanics; compatibility; meaning of wave function.

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## I. INTRODUCTION

Can the revolution of the earth be described by Schrödinger equation? If it can, it will have a great impact on the existing theoretical physics. After this method is extended to all objects, there will be the Schrödinger equation of gravitational potential energy and the Schrödinger equation applicable to all objects. Previously, people were bound by the uncertainty of microscopic particles and the non-localized realism, and the Schrödinger equation of gravitational potential energy or the Schrödinger equation describing macroscopic objects never appeared in textbooks. Since people have never tried to establish such a Schrödinger equation, it is of great significance for us to try it here. This article is an attempt, and it has been successful.

When people have to establish and apply wave mechanics or quantum mechanics, they all realize that the micro world is so different from the macro world. And recognized the notion that "the cognition, experience, rules, and theories of the macro world established by humans in theory and practice have largely failed in the micro world". So that people think that there is a huge gap between the micro world

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and the macro world. Specifically, the causality or determinism of Newtonian mechanics and classical electrodynamics, which are applicable to the macro world, are no longer applicable to the corresponding occasions in the micro world, but wave mechanics, natural randomness and uncertainty, which are applicable to the micro world, are not suitable for describing the movement changes of macro objects. Although there has been Ehrenfest Theorem [1] and it is the mission of condensed matter physics to explain macroscopic phenomena with microscopic theory. But this has not fundamentally bridged the gap between the micro-world and the macro-world. Because, influenced by Ehrenfest theorem and condensed matter physics, the relationship between macroscopic objects and microscopic particles is the relationship between quantitative change and qualitative change. Only when the Schrödinger equation is applied to the macro world without being limited by mass can the gap between the macro world and the micro world be basically bridged (the difference between them is no longer dominant).

In fact, the form and application of the Hamiltonian operator are not limited by the mass of the object, nor is it limited that the potential energy can only be electromagnetic interaction potential energy. Before this paper, people only used the Schrödinger equation on microscopic objects (if it is a microscopic object in a bound state, the binding force is only the electromagnetic interaction force, and the potential energy is the potential energy of the electromagnetic interaction force). We can't find the mathematical and logical basis for doing this, and we have to say that this is a habit formed by the bondage of ideas. There is no theoretical obstacle or mathematical logic obstacle in using gravitational potential energy in Hamiltonian operator. In this way, according to Schrödinger's method, Schrödinger equation suitable for a macro system can be established completely. The establishment of section 2 in this paper is applicable to the Schrödinger equation of gravitational potential energy of a macro-system. In the third section of this paper, the newly established Schrödinger equation applicable to macroscopic objects (including gravitational potential bound state system) is verified by using the known data of the earth's revolution.

Since the planetary model and Schrödinger equation can be used to describe an object at the same time, there is no obstacle to using the planetary model in the microscopic system to which Schrödinger equation applies. That is to say, both microscopic and macroscopic systems can use planetary model (or classical mechanics) and wave mechanics at the same time. This gives birth to the power that can make people change their existing ideas. Have readers heard of (or seen) the Schrödinger equation of the earth's revolution? If not, then the author's research work may lead a new trend. The successful establishment of Schrödinger equation of planetary motion shows that we have a theoretical basis for using wave mechanics and classical mechanics at the same time.

In reference [2-7], the author calculates atoms and molecules by using wave mechanics and planetary model at the same time, which shows that the viewpoint, theory and method that "wave mechanics and classical mechanics can be used to describe the motion system at the same time" have a wide application prospect. Wave mechanics and classical mechanics are used to describe a system at the same time, which is not limited by the mass of the system. The natural rules accumulated by the long experience in the past will not completely fail even in the micro world. This is a major conceptual revolution in the history of human development. It will also lead to a great revolution in the theory and method of basic physics.

The basic assumption in references [6-10] that "specific waves propagate along a small circle to form electrons" lays the foundation for the conclusion that "classical mechanics and quantum mechanics are compatible and can be used simultaneously on macroscopic and microscopic objects". Reference [10] derived the Schrödinger equation that can describe the Earth's orbital motion. This article is an expanded description of the content in section 3 of reference [10]. This article introduces the significance and application of the Schrödinger equation. It belongs to the category of macroscopic system Schrödinger equation and its application research progress.

The theoretical basis of this article is based on references [10,11]. After Bohr proposed the hydrogen atom planetary structure model, there was a certain connection between the old quantum theory and the classical planetary structure model. Unfortunately, it is widely believed that quantum mechanics cannot describe macroscopic systems, while classical mechanics cannot describe microscopic systems. Denying the compatibility between classical mechanics and quantum mechanics. And gradually formed this stubborn ideological concept. This closes the door to the idea of using the Schrödinger equation to describe the classical motion of macroscopic objects. Lost the opportunity to use quantum mechanics methods to describe classical planetary motion. It was not until the author of this article published multiple academic research results demonstrating the compatibility between classical mechanics and quantum mechanics [4-6], and simultaneously calculated the physical and chemical parameters (such as ionization energy, dissociation energy, bond length, etc.) of multiple atoms (ions) and small molecules using both classical mechanics and quantum mechanics methods, that the issue of compatibility between classical mechanics and quantum mechanics was reconsidered. In June 2024, the author of this article finally derived the Schrödinger equation for planetary motion [7,8].

It is the wave element electronic structure model [9-12] that resurrects the planetary model in the microscopic field. The significance of establishing and using the Schrödinger equation to describe planetary motion is not that this method is more accurate than mechanical methods, but that the establishment of this method can change people's old concepts (i.e., the idea that classical mechanics and quantum mechanics are incompatible), leading to a revolution in the interpretation system of quantum mechanics and ultimately promoting the development of quantum mechanics and even the entire physics. This research achievement is comparable to de Broglie's previous work. At that time, no one realized that the wavelength momentum relationship in the wave law also applied to particles with non-zero rest mass (especially macroscopic moving objects). However, de Broglie believed that the wavelength formula also applies to particles with zero rest mass (even moving macroscopic objects). Now, people think that the Schrödinger equation cannot be used to describe macroscopic objects, but I think it can. Our similar research work can both change people's mindset. We can wait and see if it can trigger a revolution in physics.

## II. SCHRÖDINGER EQUATION THAT CAN DESCRIBE THE REVOLUTION OF THE EARTH

In mechanics, the potential energy of phase interactions takes negative values, while the kinetic energy takes positive values. Therefore, for the interaction potential energy, it is represented by the algebraic symbol $V$, without a negative sign before $V$. If expressed using a specific calculation formula, a negative sign needs to be added before the formula. The following four expressions of the Hamiltonian operator are correct

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V,\ \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{Zee^2}{r},\ \hat{H} = -\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V\right],\ \hat{H} = -\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{GMm}{R}\right].$$

It is recommended that readers focus on checking the absolute values of the calculation results when examining the expressions and calculations in this article. This can save a lot of energy.

The stationary Schrödinger equation for hydrogen atoms is:

$$-\left[\frac{\hbar^2}{2m_e}\nabla^2 + \frac{Ze^2}{r}\right]\psi = E_e\psi. \tag{1}$$

Its one-dimensional form is:

$$-\left[\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}+\frac{Ze^2}{r}\right]\psi = E_e\psi. \tag{2}$$

Where ψ is often called wave function. One of its forms is:

$$\psi = Ae^{-i2\pi(\nu t - x/\lambda)}. \tag{3}$$

In this paper, letters with subscript e represent the physical quantity of electrons. The existing mathematical formal system and interpretation system of quantum mechanics believe that there is an insurmountable gap between the micro-world and the macro-world in terms of the laws or performances that things follow. One of the concrete expressions is that the macroscopic object cannot use wave mechanics (including Schrödinger equation), while the microscopic object does not conform to the classical mechanical theory and rules. However, as long as we analyze it carefully, it is not difficult to see that the first term of the Hamiltonian operator used to establish the Schrödinger equation is the kinetic energy operator, and the second term is the potential energy operator (also the potential energy function itself). In the solar system, the bound earth also has kinetic energy and potential energy, and it conforms to Virial theorem. Nowadays, quantum mechanics does not limit the mass of moving objects that conform to the de Broglie wave formula. We have no reason to say that we can't use the Hamiltonian operator to describe macroscopic objects. The reason why Schrödinger equation uses wave function is unknown, but it is very useful to use wave function in reality. Particles such as electrons and macroscopic objects are entities with static mass. We have no reason to say that wave functions can only be used to describe microscopic objects, but not to describe macroscopic objects. Because, for the de Broglie wave, only its wavelength is related to the quality, but there is no upper limit of the quality (that is, the macroscopic object with great mass also has the corresponding de Broglie wave).

The known de Broglie relationship is $\lambda = h/(m\upsilon)$. The group velocity of moving physical particles and/or macroscopic objects is also the velocity of the center of motion of particles and macroscopic objects. Let's redefine $\upsilon = \lambda\nu$. Among them are the frequencies of de Broglie waves of macroscopic objects or physical particles in motion. According to these two formulas, we know $h\nu = m\upsilon^2$. Considering that $E_k = \frac{1}{2}m\upsilon^2$, we have $h\nu = 2E_k$. The $\upsilon$ above is the group velocity of de Broglie waves. $\upsilon = \lambda\nu$ is a new definition proposed in this article. Only with it can we better derive the Schrödinger equation for macroscopic systems.

In Hamiltonian, the $V$ is not limited in the range of electromagnetic interaction potential energy. We also have no reason to think that the first term in Hamiltonian operator can only be applied to microscopic objects. It can be seen that we have no reason to exclude the following choices: the first term in Hamiltonian operator is also applicable to macroscopic objects, and the second term can be both electromagnetic interaction potential energy function and gravitational potential energy function. Under this premise, we can completely describe the macroscopic object by Schrödinger equation, and the Schrödinger equation used to describe the revolution of the planet is formula (4). The scope of application of Schrödinger equation, which was originally only applicable to the interaction between microscopic objects and electromagnetism, has been extended to the macro system, and it is also applicable to the four basic interaction bound state systems. This paper does not discuss the basic interaction system other than electromagnetic interaction system and gravitational interaction system.

$$-i\frac{\hbar}{2}\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi - \frac{GMm}{R}\psi = E\psi. \tag{4}$$

Where r is the distance between the earth and the sun, and e is the energy of the revolution of the earth. When the potential energy $V$=0, equation (4) is applicable to unbound macroscopic systems. Generally speaking, the logical idea of establishing equation (4) is that since there is no reason why we can't use Hamiltonian and wave function in the macro system, we might as well try to use the wave equation (Schrödinger equation) to describe the macro system in which the bound state is maintained by gravity. In the equation, E is the energy eigenvalue of a classical system bound by gravity. (4) The formula is not applicable to particles with zero rest mass, but to macroscopic systems in bound states. When $V$=0, equation (4) still applies to unconstrained macroscopic systems [see equation (8) for details]. Simply put, the logical idea for establishing equation (4) is that since there is no reason why we cannot use Hamiltonian operators and wave functions in macroscopic systems, we may try to use the wave equation (Schrödinger equation) to describe macroscopic systems where bound states are maintained by gravity (the description of unbound state systems is of course simpler). For the stationary Schrödinger equation of the Earth's revolution, there is no denominator 2 in the leftmost term of equation (4). Readers can verify it themselves. If there must be a denominator of 2, it indicates that the original Schrödinger equation is incorrect. If the denominator does not have that 2, then the table indicates that equation (8) is incorrect. This is a very serious issue that must be taken seriously. Equation (4) can be called Schrödinger-Tu equation.

When $R$ is the distance between the earth and the sun and $E$ is the energy of the earth's revolution state, equation (4) is the Schrödinger equation of the earth's revolution observed on a curved surface. When $R$ is the distance between the earth and the sun and $E$ is the energy of the earth's revolution state, equation (4) is the Schrödinger equation of the earth's revolution observed on a curved surface. In this way, $R$ is also known. Equation (4) need not be solved, but only the known constants can be substituted into Equation (4) to calculate the kinetic energy and potential energy of the earth's revolution and the total energy $E$ of the earth's revolution. Using the three-dimensional form of Equation (4) to calculate the energy eigenvalues of the earth is unnecessary.

## III. VERIFICATION OF THE SCHRÖDINGER EQUATION OF THE EARTH'S REVOLUTION

Considering de Broglie relation $\lambda=h/p=h/mv$, classical momentum formula $p=mv$, $\upsilon=\lambda\nu$, $\hbar=h/2\pi$ and Eq. (2), the value of the first term on the left of the equal sign of Eq. (4) is:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi=\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2\psi=\frac{m\upsilon^2}{2}\psi. \tag{5}$$

The coefficient $\frac{m\upsilon^2}{2}$ is the sum of kinetic energy, when the earth revolves, and $\upsilon$ is the revolution speed of the earth. According to the relationship that the centripetal force of the earth's revolution is equal to the attraction of the sun to the earth, m= rυ/g. Therefore, the value of the second item on the left of equation (4) is:

$$-\frac{GMm}{R}\psi=-m\upsilon^2\psi. \tag{6}$$

Substituting the expressions (5) and (6) into the expression (3) and eliminating ψ, we get the energy eigenvalue of the system:

$$E=\frac{m\upsilon^2}{2}-m\upsilon^2=-\frac{m\upsilon^2}{2}. \tag{7}$$

Among them, $-mv^2$ is the potential energy of the earth's bound motion, and it is exactly twice the kinetic energy of the earth's revolution (which shows that the Schrödinger equation of the earth's revolution guarantees the establishment of the Virial theorem). This is exactly the same as the result calculated according to classical mechanics. In this way, as shown in the second equation of equation (7), the earth energy e in equation (3) is equal to $-\frac{m\upsilon^2}{2}$. This is the classical expression of the energy of the earth's revolution, and it is the energy eigenvalue solution of equation (4). Obviously, we have proved that equation (3) holds, and we can use Schrödinger equation to describe the revolution of the earth.

## IV. SCHRÖDINGER EQUATION OF PLANETARY MODEL OF HYDROGEN ATOM AND ITS VERIFICATION

Our previous conclusion was that both the Schrödinger equation and the planetary model can be used simultaneously or separately to describe macroscopic objects such as Earth, there is no reason to restrict the simultaneous use of the Schrödinger equation and classical mechanical models (of which the planetary model is one) to describe microscopic systems such as hydrogen atoms. We can use both the planetary model and the Schrödinger equation to describe the hydrogen atom. We still choose to observe hydrogen atoms on a curved surface (using a planetary model, the orbital motion of electrons is similar to the orbital motion of planets. We can observe hydrogen atoms in Riemannian space). Use equation (2) to describe the orbital motion of electrons in hydrogen atoms, where the first term on the left side of the equation corresponds to the kinetic energy $E_k$ of the electron orbital motion. The Schrödinger equation in atoms also conforms to equation (5). No way, $m_e$ is the mass of the electron, and $m\upsilon^2/2$ is the kinetic energy of the electron in the hydrogen atom (where υ is the group velocity of the electron, i.e. the group velocity of the electron's de Broglie wave). When V=0, equation (4) becomes

$$i\frac{\hbar}{2}\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi=\frac{1}{2}m\upsilon^2\psi=E_k\psi. \quad (8)$$

In the equation, $\frac{m_e\upsilon^2}{2}$ is the kinetic energy of the electrons in the hydrogen atom (where $\upsilon$ is the group velocity, *i.e.* the group velocity of the electron's de Broglie wave) . The potential energy of an electron is $-\frac{Ze^2}{r}$. When $r$ is constant value $a_0$, When the potential energy of the system is zero, it is not simply to eliminate the potential energy in equation (4), but to consider the properties of the calculated energy, that is, to take into account the sign of the calculated result. The result calculated according to equation (8) is kinetic energy, while equation (4) calculates the energy eigenvalue of the system. This is the reason why the first term in equation (8) does not have a negative sign. The ground state energy eigenvalue of the hydrogen atom solved according to equation (2) is $-\frac{Ze^2}{a_0}$. According to the force law of uniform circular motion of bound state:

$$m\upsilon^2=\frac{Ze^2}{a_0},\ \ \upsilon=\sqrt{\frac{Ze^2}{a_0m_e}}. \quad (9)$$

According to the two relationships of $a_0=\frac{h}{m_ec\alpha}$ *and* $\alpha=\frac{e^2}{hc}$, by eliminating $a_0$ and $e^2$ in equation (9), we can obtain $\upsilon=\alpha c$. Here, $\upsilon=\alpha c$ is not only the planetary orbital velocity of ground-state hydrogen atom, but also the group velocity of de Broglie wave of electrons in ground-state hydrogen atom. Although equation (9) contains the solution of wave mechanics equation, it conforms to classical mechanics theory. This proves once again that classical mechanics and quantum mechanics can be compatible (The previous proof is that they are compatible in the macro field, and this section proves that they are compatible in the micro field).

## V. EXPLORATION ON THE ESSENCE OF WAVE FUNCTION $\psi$

Schrödinger used the wave function (4) in those days, but he didn't know its true meaning, just regarded it as a part of mathematical tools. After him, no one clarified the essence of wave function $\psi$. There is still an unsolved mystery about the nature of matter wave. The first formula in the Eq.(9) accords with the classical mechanical theory.

Macroscopic objects moving in a straight line can also be described by Schrödinger equation. For example, the Schrödinger equation of a train moving in a straight line at the speed $\upsilon_t$ is:

$$-\frac{\hbar^2}{2m_t}\frac{\partial^2}{\partial x^2}\psi = E_t\psi. \tag{10}$$

Solving this equation is very simple, just need to find the partial derivative, and use the formula (4) and the de Broglie relation: $\lambda_t$=h/$p_t$= h/($m_t\upsilon_t$). The solution of equation (10) is $E_t = E_k = \frac{1}{2}m_t\upsilon_t^2$. This solution shows that the energy eigenvalue solution of an object in linear inertial motion is the non-relativistic kinetic energy of this object. Comparing this solution with the velocity-frequency relation $\upsilon_t=\lambda_t\nu_t$ of monochromatic wave, we can get:

$$E_t=\frac{1}{2}h\nu_t. \tag{11}$$

Equation (11) shows that the energy of the object's matter wave is twice its kinetic energy. This does not conform to the law of real monochromatic waves. It shows that the matter wave of an object cannot be a complete wave. This result is obviously beneficial to explore the essence of the matter wave.

Since the major obstacle of "describing macroscopic objects by wave mechanics" has been removed by establishing the available Schrödinger equation of planetary motion, we can calculate how many deuterium-like atoms are contained in the earth by using the wave mechanics method of microscopic system (deuterium atoms are calculated as simulated cells that make up macroscopic objects). The result must be very interesting.

The angular momentum of an object moving in a circle is L = r× p. Replacing P with the momentum operator $\vec{L} = \vec{R} \times \vec{p}$, we have obtained the operator for the angular momentum of electron orbitals in hydrogen atoms.

$$\hat{L}_N = -\, iN\hbar R\frac{\partial}{\partial x}. \tag{12}$$

Adding footnote N only emphasizes the description of N unit objects. By applying it to Apollo function $\psi$ [see formula 3], we can get

$$-\, i\hbar NR\frac{\partial}{\partial x}\psi = \hbar N\frac{R}{r}\psi. \tag{13}$$

Eliminate $\psi$ in the above formula, and the angular momentum formula of the macroscopic object or the simulated composite particle with hydrogen atom as the unit to do the circular motion of the bound state.

$$L_N= N\hbar\frac{R}{r}. \tag{14}$$

$L_N=Rmv=2.658\times10^{37}$ (m·kg·s). The mass m of the earth is $5.965\times10^{24}$kg. The distance r between the sun and the earth is 149597870 kilometers (1.496 million kilometers). The average revolution linear velocity of the earth $v$ is: 29.783 km/s (107,220 km/h). For describing the earth's revolution with wave function, r in wave function $\psi$ and (14) is equivalent to the radius $a_o$ of hydrogen atom ($a_o$ = 5.2918× $10^{-11}$ meters). ħ=1.054571726×$10^{-34}$ J·s. We substitute these numbers into equation (14) and we can get

$$N= \frac{La_0}{\hbar R} = 8.734\times10^{50}. \quad (15)$$

It is not difficult to see that the product of the mass of N and the deuterium atom as the mass cell of macroscopic substances (including simulated composite particles with deuterium atom as the smallest unit and other compounds) is the mass of the earth. The mass of deuterium atom is 3.3688×$10^{-27}$kg. In this way, the mass of the earth calculated by wave mechanics method is:

$$m_{earth}=N\times3.3688\times10^{-27}kg=2.942\times10^{24}\ kg. \quad (16)$$

This calculated value is of the same order of magnitude as the known Earth mass value of 5.965×$10^{24}$kg . There are two main sources of error: first, the molecular structure of the earth is complex, and deuterium atoms cannot be accurately used as simulated mass cells of the earth; Second, other energies contained in the earth and the sum of binding energies in molecules are not considered.

The mass of the earth is calculated by wave mechanics, which proves that wave mechanics is effective in dealing with planetary motion. The results of this analysis can at least remind us why we can use the wave function $\psi$ when describing macroscopic and microscopic objects. It also supports the theory of wave element material structure proposed in references [3-6].

## VI. CONCLUSION

The revolution of the Earth is absolutely on a plane, and within a certain period of time, R is a certain value. If we assume that the ball moves randomly in three-dimensional space (which is an uncertain motion without an orbit) and solve an equation similar to Eq. (1), there will definitely be extra roots. If the state of electrons in hydrogen atoms is determined, then using the three-dimensional Schrödinger equation for calculations can also lead to unrealistic solutions. The research results of this article suggest this possibility.

A conclusion of Section 4 is that the classical mechanical method and the quantum mechanical method are compatible and can be used at the same time, whether describing macroscopic objects or microscopic objects. For convenience, we call this conclusion conclusion 1. The use of classical mechanical methods means that the described object is deterministic, realistic and localized, and conforms to determinism. It can be seen that conclusion 1 cannot absolutely deny that micro-objects are also deterministic, localized and causal (only in a narrow range or under certain conditions can people show uncertainty, non-localization, unreality and indecision). This is an important inference according to conclusion 1.

Conclusion 1 and its inference show that the gap between macro-system and micro-system can be eliminated or reduced. Wave mechanics and classical mechanics can be used to describe objects from micro to macro at the same time, which can simplify the calculation process. Determinism, localization, realism and determinism cannot be completely denied in the microscopic system. This obviously has great influence on the interpretation system of quantum mechanics. The obstacles to establishing localized real quantum mechanics are also much smaller.

At that time, Schrödinger did not explain the reason why he used the wave function of formula (4) in the Schrödinger equation of hydrogen atom [that is, he did not specify the meaning of formula (4), but only used it as a tool]. We can be sure that the revolution of the earth is definitely not a wave like Eq. (4). This paper proves that "the correct result can be obtained by using equation (4) when describing the revolution of the earth". This result further strengthens the concept that "wave function ψ is a tool in wave mechanics". Unless both microscopic and macroscopic objects are made of waves. If there are no particles but waves in the constituent elements of matter, we can start to establish the theory of wave element material structure.

It is recognized that "human beings have not yet loved the combination of relativity (or gravity theory) and quantum mechanics." This paper proves that we can use Newton's mechanics and quantum mechanics at the same time (that is, Schrödinger equation, the basic equation of Newton's gravitational interaction potential energy and wave mechanics) to describe the motion and microscopic system of celestial bodies. This is the compatibility and combination of Newton's gravity theory and quantum mechanics to a certain extent (although gravity is not quantized, it is combined in another way).

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The existing interpretation system of quantum mechanics is determined by the principle of uncertainty or the uncertainty of particles. The factors that affect the uncertainty principle will ultimately affect the existing interpretation system of quantum mechanics. The significance of the Schrödinger-Tu equation lies not in using it to obtain more accurate computational results than Newtonian mechanics, but in changing the old notion that the gap between classical mechanics describing macroscopic objects and quantum mechanics describing microscopic objects is insurmountable. Another function is to prove that the concepts of "electron cloud" and "probability density" are meaningless by solving the Tu equation. Because we believe in the determinacy of macroscopic objects, it is impossible to obtain the concepts of "Earth clouds" and "probability density of the Earth's center of mass in the space of the solar system" by solving Schrödinger-Tu equation. That is to say, it proves that the concepts of "electron cloud" and "probability density" are derived by pre-affirming "uncertainty". This has a significant impact on the interpretation system of quantum mechanics. Readers who do not understand, please continue reading. To obtain the concepts of "Earth's center of mass cloud" and "probability density of Earth's appearance in the space of the solar system" by solving the Tu equation, it is necessary to first assume that the motion of the Earth in the solar system conforms to the "uncertainty principle" (the specific idea is to assume that the distance between the Earth and the Sun is uncertain and unknown, only knowing that this distance is from 0 to infinity). Otherwise, we won't get those two concepts no matter what. That is to say, those two concepts originate from the uncertainty principle, not from the solution of the wave equation. In other words, those two abnormal concepts are products of a priori theory, rather than products of mathematical logic.

## VII. DISCLAIMER

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*Competing Interests*

Author has declared that no competing interests exist.

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# Can Comparing Traditional and Green Infrastructure Promote Ecosystems Restoration? Case Study of Three Restoration Assessments in Cameroon

*Peter Mbile, Fadcre, Lylliane Elomo, Fadcre, & Yaya Fodoue*

## ABSTRACT

Much of Africa's forest ecosystems heritage can be found in Cameroon. The country has since ratified numerous Conventions and enacted laws to protect and valorize these connecting ecosystems benefiting local people and global climate. These forest and non-forest ecosystems constitute Cameroon's green infrastructure today. However, due to anthropogenic and natural processes, these ecosystems face degradation, thereby weakening their superstructure, diminishing their services value; threatening livelihoods, and contributing to climate change.

In this paper, we draw parallels between green infrastructure and traditional (hard) infrastructure, in order to bring  to ecosystem restoration,comparable maintenance mindset, historically reserved for hard infrastructure. We use the prisms of three ecosystems assessments for restoration, as case studies. These are; (i) the northern savannah, (ii) Sanaga-Kadey watershed and (iii) the forest transition zones of Cameroon. By analyzing some common parameters across these ecosystems, including (i) land tenure, (ii) multifunctionality, (iii) climate resilience, (iv) critical resource use efficiency, (v) carbon neutrality, (vi) connectivity, (vii) stakeholder engagement, (viii) social inclusivity and (ix) maintenance-friendliness, we simultaneously make a case for adopting analogous maintenance mindsets towards securing and re-building Cameroon's threatened green infrastructure

*Keywords:* green infrastructure, ecosystem restoration, inclusiveness, Cameroon.

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# Can Comparing Traditional and Green Infrastructure Promote Ecosystems Restoration? Case Study of Three Restoration Assessments in Cameroon

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## ABSTRACT

*Much of Africa's forest ecosystems heritage can be found in Cameroon. The country has since ratified numerous Conventions and enacted laws to protect and valorize these connecting ecosystems benefiting local people and global climate. These forest and non-forest ecosystems constitute Cameroon's green infrastructure today. However, due to anthropogenic and natural processes, these ecosystems face degradation, thereby weakening their superstructure, diminishing their services value; threatening livelihoods, and contributing to climate change.*

*In this paper, we draw parallels between green infrastructure and traditional (hard) infrastructure, in order to bring to ecosystem restoration,comparable maintenance mindset, historically reserved for hard infrastructure. We use the prisms of three ecosystems assessments for restoration, as case studies. These are; (i) the northern savannah, (ii) Sanaga-Kadey watershed and (iii) the forest transition zones of Cameroon. By analyzing some common parameters across these ecosystems, including (i) land tenure, (ii) multifunctionality, (iii) climate resilience, (iv) critical resource use efficiency, (v) carbon neutrality, (vi) connectivity, (vii) stakeholder engagement, (viii) social inclusivity and (ix) maintenance-friendliness, we simultaneously make a case for adopting analogous maintenance mindsets towards securing and re-building Cameroon's threatened green infrastructure.*

*Keywords:* green infrastructure, ecosystem restoration, inclusiveness, Cameroon.



## I. INTRODUCTION

### 1.1. *Traditional versus green infrastructure*

Traditional infrastructure[1] refers to the fundamental physical and organizational structures and facilities needed for the operation of a society, enterprise, or system. It includes essential services such as transportation systems (roads, bridges, airports), utilities (water supply, electricity, telecommunications), buildings (schools, hospitals, offices), and other key facilities necessary for the functioning of a community or business. Such infrastructure plays a critical role in supporting economic development, social well-being, and overall quality of life. Such traditional infrastructure can also have negative environmental impacts, such as increased pollution, habitat destruction, and water runoff[2].

While traditional infrastructure is primarily concerned with meeting human needs through built structures, green infrastructure emphasizes the integration of natural systems to provide multiple

[1] . "Green Infrastructure." Environmental Protection Agency, www.epa.gov/green-infrastructure/what-green-infrastructure
[2] United States Department of Agriculture. "Green Infrastructure." Natural Resources Conservation Service, www.nrcs.usda.gov/wps/portal/nrcs/main/national/plantsanimals

benefits for both people and the environment[3]. Principally, green infrastructure focuses on incorporating natural elements and processes to provide a range of ecosystem services while also promoting sustainability and resilience. Green infrastructure typically uses nature-based solutions [1,2,3] to mimic natural processes, enhance biodiversity, improve air and water quality, reduce heat and island effect in urban areas, to create more livable and healthy built-up communities.

At the scale of a country, green infrastructure refers to the network of natural and semi-natural areas that have evolved (modified and managed) to deliver multiple ecological, social, and economic benefits. Ideally, large-scale green infrastructure comprises interconnected green systems, such as national parks, wetlands and hydrological systems, high value forests, watersheds, agricultural and rangelands, etc. They provide ecosystem services, improve air and water quality, protect wildlife habitats and enhance biodiversity. Green infrastructure facilitates seed dissemination, agricultural crop pollination, soil fertility and regeneration; promoting human health, well-being and economic development.

Similar to the rehabilitation and maintenance needs, historically restricted to the built, urban and or traditional infrastructure, concerns about the state of green infrastructure are rising, and restoring degrading, green interconnected ecosystems [4] and landscapes, is today of major national and global concern, yet nowhere near the maintenance mindset reserved for hard infrastructure.

### *1.2 Cameroon's green infrastructure endowments*

Cameroon is located in Central Africa and shares borders with Nigeria, the Central African Republic, Chad, the Republic of the Congo, Gabon and Equatorial Guinea. With a population of approximately 25.7 million people as of 2020, Cameroon has a mixed economy featuring state-owned and private enterprises. Food and export crop agriculture drives the economy, accounting for 22% of GDP with export crops like cocoa, coffee, banana, and palm oil. The country also has oil and gas reserves and manufacturing sectors for textiles, food processing, and construction materials. Services contribute 50% to GDP, with a per capita income of $2,300. Challenges include corruption, infrastructure limitations, and security concerns [5]

Cameroon's green infrastructure is represented by its forests repartitioned into permanent and non-permanent estates under the Cameroon Forestry Law of 1994. The country's permanent forest estate is estimated at 15,7 million hectares, and the non-permanent estates at 6.9 million hectares. This is out of a total forest area estimated at 22.5 million hectares [6].

By 2013 Cameroon[4] had officially partitioned the national territory into five (5) Agroecological zones; two forest zones; (i) a coastal with mono-modal rainforest regime (ii), bimodal hinterland forests; (iii) a high plateau, (iv) high savannah and (v) sudano-sahel zones. Each zone possesses slightly peculiar, dominant vegetation (main component of green infrastructure) with elevation (e.g., montane forests), nearness to sea (mangrove systems), low humidity and or low soil organic matter (sudano-sahel ecosystems) being additional factors influencing the classification and functioning of ecosystems or biomes. By 2020 there were nineteen protected areas in Cameroon with at least one in each of the five agro-ecological zones. The 1994 Forestry Law and one of its texts of Application – the 1996 National Environmental Management Plan (NEMP), stipulates a network of protected areas, to ensure that a viable sample of each type of ecosystem or biome is represented/protected, continues to provide its unique ecosystems functions, and to the extent possible, is connected to other natural systems. The presence of five agro-ecological zones, including sub biomes hasn't only earned Cameroon the title of

[3] Benedict, Mark A., and Edward T. McMahon. Green Infrastructure: Linking Landscapes and Communities. Island Press, 2006

[4] A central African

"Africa in miniature", but Cameroon is also famously, a meeting point of two major continental biomes – the west African upper Guinean and the central African-Congolian forest systems.

### *1.3 Cameroon's commitments to secure her green infrastructure*

To solidify her commitment to secure her green infrastructure heritage, Cameroon has signed and ratified several major multilateral environmental agreements related to forests, biodiversity, and climate change. Some of the key agreements and strategies include:

- Convention on International Trade in Endangered Species of Wild Fauna and Flora (CITES): signed since 1983 and implemented to regulate the trade in endangered species.
- Convention on Biological Diversity (CBD) signed in 1992 ratified in 1994, committing to conserve and sustainably use her biodiversity.
- As required by the CBD Cameroon developed a National Biodiversity Strategy and Action Plan (NBSAP) in 2001 and updated, in 2015.
- The Kunming – Montreal Global Biodiversity Framework, being an offshoot of the CBD was adopted by Cameroon in 2022, guiding the revision of her NBSAP in 2023.
- Cameroon signed the United Nations Framework Convention on Climate Change (UNFCCC) in 1992 and ratified it in 1994.
- Cameroon adopted the Declaration of the Summit of Heads of State of Central African Countries on the Conservation and Sustainable Management of Tropical Forests of 1999.
- Cameroon has been a party to the United Nations Forests for People Programme (UNFP) since its inception in 2000.
- Cameroon ratified the Central African Forest Commission (COMIFAC) Convergence Plan in February 2005 on harmonized, sustainable management of Congo Basin forests.
- Cameroon is a signatory to the African Forest Landscape Restoration Initiative (AFR100): a continental initiative launched in 2015 to restore 100 million hectares of degraded landscapes across Africa by 2030.
- Cameroon has developed a National Development Strategy with a 2030 horizon (SND30) that includes sustainable management of the country's forest and other green infrastructure.

Despite these commitments to address environmental issues such as climate change, biodiversity loss, and deforestation, and demonstrated willingness to work with global actors to achieve common goals, Cameroon's green infrastructure heritage is still confronted with degradation.

For instance, from 2001 to 2023, Cameroon lost 2.05 million ha of tree cover, equivalent to a 6.5% decrease in tree cover since 2000, and equivalent to 1.23 Gt of $CO_2$eq of emissions (GFW, 2023). Amongst Cameroon's commitments, her pledge through the AFR100 and Bonn Challenge to bring 12, 062,768 hectares of degraded landscapes under restoration by 2030 stands out as particularly relevant towards rebuilding and maintaining the country's green infrastructure.

To update the country's environmental management commitments, a new forest and wildlife law no. 2024 008 of 24 July 2024 has been adopted. It builds on the 1994 framework laws and focuses on biodiversity conservation, climate change and the sustainable development goals; community rights; a landscape approach to environmental management, landscape restoration, use of advanced monitoring systems, stiffer law enforcement regulations for critically endangered species, greater transparency and much stronger alignment with international conventions.

### 1.4 Cross-cutting issues on the nexus of hard and green infrastructure

Like traditional (hard) infrastructure, development of green infrastructure (tree-based ecosystems), has some important prerequisites or requirements such as; land tenure, stakeholder engagement, social inclusivity and maintenance-friendliness. Any infrastructure is also expected to possess certain characteristics enabling them to deliver services in a particular manner, including; multifunctionality, climate resilience, critical resource use efficiency, carbon neutrality and connectivity, among others. Through these commonalities, green and traditional infrastructure appear to seek similar goals – that of achieving resilience, by reducing the risk of failure, improving system performance, and ensuring that infrastructure can adapt to changing conditions over time.

From a socio-cultural perspective, one overriding requirement for green infrastructure in Cameroon (equally important to traditional, hard infrastructure) – is tenure, and this deserves some emphasis.

The tenure status across all the areas of opportunities for green infrastructure development are also a reflection of Cameroon's 1974 land tenure laws. A number of exceptions to this Law can exist under specific conditions. Although rarely invoked, such exceptions in Cameroon may include land title deeds listed under the pre-independence *Grundbuch*[5] in former west Cameroon. Even with such lands, either retroceded to traditional authorities or held in trust by customary entities such as lands within Lamidats[6] in northern Cameroon, only a land title transfers full rights of ownership from one moral entity to another; the rest being considered as lands in the *National Domain*[7]. The tenure status of such lands must often be ascertained prior to developing either green or hard infrastructure.

Furthermore, in 1994, forested lands (as defined by Law n°94/01 of January 20, 1994) came under the jurisdiction of the forest and wildlife laws, and were demarcated as Permanent (PFE) and Non-Permanent Forest Estates (NPFE). Whereas PFE are "the private property" of the State (comprising protected areas, timber concessions, council forests, etc.), NPFE are defined to include agroforestry zones, comprising *inter alia,* community forests, and private forests, where individuals and communities can exercise ownership and control rights under certain agreements with the supervising authorities (e.g. in the case of Community forests). Given that forests have been defined to cover the national territory, such estates can occur everywhere, including in the northern regions, not traditionally considered to be "a forest zone". Tenure statuses therefore, have implications for all types of green infrastructure development to be implemented, and who the stakeholders can be.

## II. CASE STUDIES OF GREEN INFRASTRUCTURE OPPORTUNITIES ASSESSMENTS

To illustrate issues pertaining to green infrastructure development in Cameroon, three case studies are used to articulate prerequisites and requirements of successful green infrastructure development. These case studies illustrate salient issues to consider in the event of a shift in mindset towards a stronger maintenance culture for green infrastructure, similar to that hitherto reserved for hard, traditional infrastructure.

For a consistent ecological cognition rather than a hard infrastructural one, the expression "ecosystem restoration" will be used to mean "rebuilding green infrastructure", meanwhile, the principal micro ecosystems examined in this paper are tree-based systems.

[5] Refers to lands held/contested as being under customary trust (especially by the Bakweri ethnic group) and believed to have been excluded from the 1974 Land Tenure Ordinance of Cameroon that stipulates that, all land is the property of the State.
[6] A Lamidat is a traditional Muslim chiefdom in northern Cameroon (currently the Far North, North and Adamaoua regions).
[7] An attribution meaning Sovereign lands or "property of the State" of Cameroon.

The three ecosystem restoration case studies are therefore, (i) tree savannah, northern region; (ii) the Sanaga – Kadey watershed, in the eastern region, and (iii) a forest-savannah transition zone, Centre region, all in Cameroon.

### *2.1 Northern tree savannah ecosystem restoration (ER) assessment – overview*

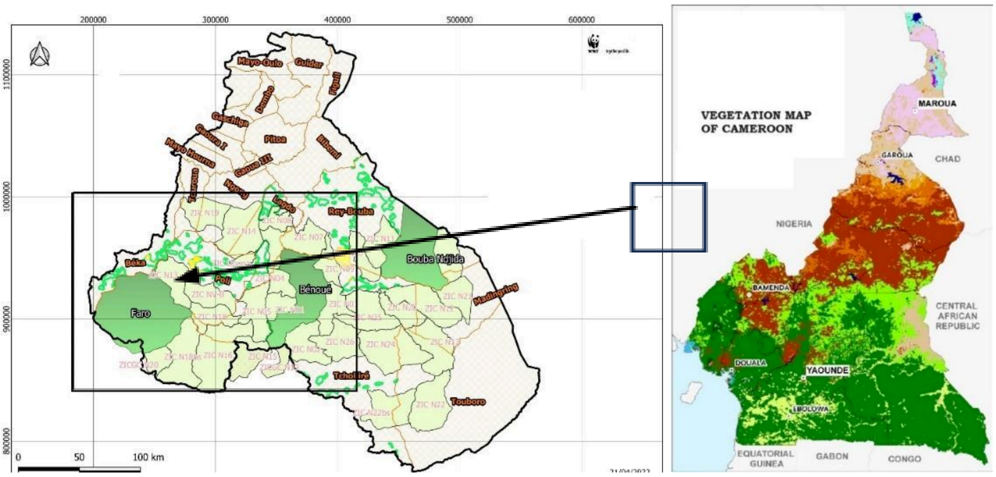


*Figure 2:* Northern savannah assessment and opportunities for ER

### *2.1.1 Survey of restoration opportunities in the northern savannah*

In June of 2022 a sub national Restoration Opportunities Assessment Methodology [4], was completed here by WWF Cameroon [7]. The assessment identified 26,029 ha to be brought under ecosystem restoration. 89.59 % of this area are agricultural lands, 10.11 % agroforestry and 0.3% appropriate for woodlands. The area is densely populated, water-stressed with river banks suffering strong erosion; low soil organic matter and water retention potential; and characterized by impoverished and degraded agricultural soils (leached and low in soil nutrients). Indigenous tree species; *Pterocarpus erinaceus, Afzelia Africana* and *Kiguelia Africana,* are selectively overexploited here for fuel, timber, furniture and fodder.

The ecological needs highlighted by this case study pertain to; multifunctionality, climate resilience and critical resource use efficiency aspects of ecosystems restoration (or green infrastructure development).

### *2.1.2 The Sanaga – Kadey watershed assessment – overview*

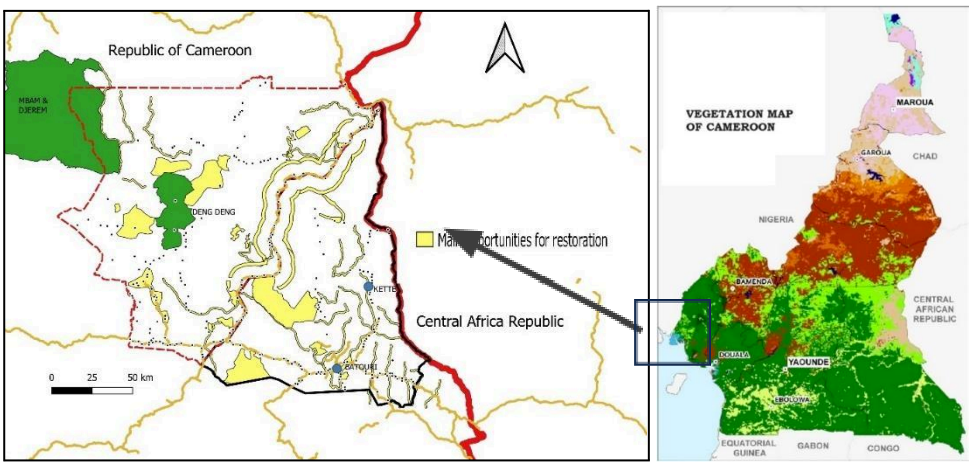


*Figure 3:* Sanaga - Kadey watershed and opportunities for restoration

### 2.1.3 Survey of restoration opportunities in the Sanaga – Kadey watershed

In June of 2023 a second sub national Restoration Opportunities Assessment Methodology [4], was completed by WWF Cameroon [8] in the Sanaga-Kadey watershed and identified 426 904 ha of ecosystems restoration opportunity. 98 942 ha of this is in communal forests (state forests); 10 070 ha of abandoned mines within the national domain; 65 062 ha of micro zoned areas also in the national domain; 107 001 of gallery (riparian) forests within the national domain and 145 829 ha of degraded agroforestry areas within the national domain and within community areas. With non-uniform population distribution, community areas including riparian forests are densely settled, whereas the national domains and abandoned mines are sparsely populated. Agricultural areas are dominated by annual crops (cereals and tubers); and agroforests of indigenous multipurpose trees species; *Irvingia gabonensis, Ricinodendron heudelotii* and *Trichoscypha acuminata;* medicinal species; *Baillonella toxisperma, Enantia chlorantha* and *Garcinia cola;* and by timber species, *Diospyros spp. Triplochiton scleroxylon, Afzelia bipidensis* and *Entandophragma cylindricum*. Most sensitive degradation drivers; small scale agriculture and fuelwood extraction impacts riparian forests, while artisanal mining impacts land cover and water quality.

The ecological needs highlighted by this case study pertain to; multifunctionality, critical resource use efficiency and connectivity aspects of green infrastructure development.

### 2.2 The forest-savannah transition zone agroforestry assessment: overview


*Figure 4:* Forest transition agroforestry landscape opportunities for restoration

### 2.2.3 Restoration practice in the "Ndong community" (see Mbile & Elomo, 2024)

Between 2003 and 2004, the World Agroforestry Centre (ICRAF) transferred 28,000 trees of *Irvingia wombolu*, an indigenous fruit tree, into the NDONG community, Centre region, under the tree domestication programme that began in the 90s [9]. This fragmented forest transition zone just outside Yaoundé (Cameroon's capital City) is a forest degradation hotspot [10] and community livelihoods here have partly depended, and continue to do so, on consumption and sales of products from indigenous tree species including *Ricinodendron heudelotii*, *Monodora myristica*, as well as exotic economic trees like Avocado, Citruses and Oil palm. The participants of the programme included men, youth and a women's group – "MERUNGA". This restoration programme has been monitored for over 20 years and the results published [11].

The ecological lessons learned in this case study involve indigenous tree domestication through the displacement of viable planting materials for ecosystem restoration, and are particularly relevant to;

multifunctionality, social inclusiveness and maintenance-friendly aspects of green ecosystems restoration (or green infrastructure development).

## III. DISCUSSION OF THE CASE STUDIES IN THE CONTEXT OF GREEN INFRASTRUCTURE DEVELOPMENT

### *3.1 Building resilience*

To the same extent that the problems of ecosystem degradation are numerous and interconnected, so too should the interventions to resolve them, are of a multifunctional nature.

The northern regions of Cameroon are amongst the most climate vulnerable with longer hours of insolation, high seasonal temperatures and short-lived, but torrential rains. This facilitates soil crusting, and formation of hardpans leading to low water infiltration and potentially higher volumes of surface water. Low soil organic matter, low soil carbon holding capacity and high runoffs exacerbate erosion and soil loss (Figure 5). The low overall forest cover, high exposure enhances vulnerability to strong winds and dust storms causing agricultural land degradation (Figure 6).

*Figure 5:* Dry river beds, eroded river banks and degraded gallery forests in the Faro National Park, landscape, North region

*Figure 6:* Agricultural fields in the north region suffer from low organic matter, tree cover, high insolation, desertification and high exposure

Moving on to the Sanaga-Kadey watershed on the eastern forest transition zones of Cameroon, the population is more dispersed compared to the north region. However, this ecosystem faces similar pressures on riparian forests (Figure 7) that protect rivers and streams.

Whereas in the drier northern savannah, there is strong preference for arable agricultural land (89.59 % of assessed opportunities) with fewer trees, rampant wildfires used to prepare agricultural and pastoral lands, including unsustainable extraction of fuelwood drives tree-loss. This form of degradation is also common here in the Sanaga-Kadey watershed (Figure 3 and Figure 7).

In both cases, tree loss on these vulnerable ecosystems generally increases the likelihood of hazards from extreme weather events, like storms, flooding, causing loss of top soil, siltation of waterways, while extreme droughts, exposure and strong winds drive uncontrolled wildfires, decimating what little vegetation is left, thereby reenforcing the degradation cycle.

*Figure 7:* Riparian forest degradation on the banks of the Bindiki watercourse (Garoua - Boulai) Sanaga – Kadey watershed, east region

As human populations grow, there is increasing need for more, often low-productivity farmlands and so more forests are converted to farmlands.

The recommended restoration opportunities for both the northern savannah and the Sanaga-Kadey watershed thus emphasize how to enhance the multifunctionality of the ecosystems by sustainably managing and conserving critical resources like water, and tree systems diversity (abundance and evenness) for increased resilience. Such ecosystems restoration opportunities were therefore most strongly recommended for sensitive, high value and easy-to-manage areas of both northern savannah and Sanaga-Kadey watersheds, such as managed farmlands, community home gardens, and relatively smaller areas such as river banks (or gallery forests), community forests and sacred forests (making 11% of total assessed area).

Development of agroforestry systems in both the northern savannah region and the Sanaga-Kadey watershed by planting a range of available, multipurpose trees species (fruit, timber, fuel wood, soil improvers) is one solution. For agricultural land, application of bio regenerating products like Biochar (a carbon-rich material produced by partial oxidation- pyrolysis, at ≤700 °C in the absence or limited supply of oxygen, using organic materials such as forestry and agricultural wastes as substrate (32, 33) to improve soil water holding capacity, cation exchange capacity and overall fertility are also recommended.

### *3.2 Enhancing critical resources use efficiencies*

Critical resource use efficiency is an important ecosystem restoration strategy in multifunctional landscapes. Such critical resources include soil nutrients holding capacity and species (e.g. trees) biodiversity with a direct correlation with system resilience [12, 13, 14]. Due to relatively low growth rates of especially indigenous species, and the harsh soil biophysical factors of low organic matter and hard-pans, the recommended technology to rehabilitate such soils in the north savannah region is the relatively costly Revitech[8] [15, 16] estimated to reach $US 1,500 per hectare. This technology, although tested and viable in the targeted context, is costly, and well beyond the financial wherewithal of the relatively poor communities. Its low adoption rates to-date, despite it proven effectiveness is largely due to cost.

The preference for slow-growing indigenous species, for biodiversity and resilience, can actually add to the ecosystem restoration costs per hectare, as it takes longer for indigenous trees to grow, and establish, if at all. As a consequence, and based on lessons from the application of vegetation successions in restoration [16,17, 18,] the application of Revitech on hardpans is still recommended despite costs. The strategy involves the use of fast-growing exotic species (e.g. Acasia) as part of a vegetation succession, creating enabling micro-conditions, which then allows slow growing indigenous species to establish as part of assisted natural regeneration.

Another recommended strategy to restore hard-pans (including abandoned mines where topsoil has been removed), support soil fertility (a critical resource), assist natural regeneration and grow biodiversity, is through the use of biochar through private sector participation. In the northern savannah, biochar production is envisaged through pyrolysis of rice husks, a waste product of the rice-producing company SEMRY (*YAGOUA Rice Expansion and Modernization Company*).

Similarly, restoration interventions to enhance soil carbon and organic matter in the Sanaga – Kadey watershed are expected to source forestry waste produced from lumbering in communal forests or supplied under contract from private timber companies like PALISCO LLC (a private forestry company in the southeastern forest) operating just south of the watershed. The coordinated role of the private sector is further discussed below under private sector engagement.

Furthermore, the northern savannah degraded areas are in a transhumance zone, prone to farmer-grazer conflicts and frequent bush fires. An interesting (bitter-sweet) relationship exists here between farmers and graziers. The sweet part involves agricultural land fertilization via cow dung. Meanwhile, the bitter parts involve deliberate fires set by pastoralists to stimulate re-sprouting of fresh fodder, crop and saplings raiding by livestock. Managing this bitter-sweet scenario, prevalent in both the northern savannah and Sanaga-Kadey watershed requires (costly) effective and reliable monitoring missions (passive restoration) to manage conflicts and ensure survival of species undergoing natural regeneration.

The restoration assessments [4] for both northern savannah and Sanaga-Kadey watershed, thus recommended species with multiple uses (for soil fertility, medicinal, food values, fodder value, fuel-wood value, etc.), and with strong economic potential (e.g., Cashew nut) to serve as incentive for engaging local communities [20].

### *3.3 Carbon capture and climate action*

A significant amount of biochar use is envisaged in these recommended ecosystems restoration processes. Biochar is a negative emissions technology, identified by the Intergovernmental Panel on Climate Change (IPCC) as effective in mitigating climate change and achieving net-zero emissions [21]. Biochar increases the soil's capacity to retain, absorb carbon and support natural carbon sinks. It improves soil structure and health (soil organic matter, soil carbon content, microbial and fungal

[8] Revitech increases soil organic matter and thereby nutrient and moisture holding capacity of soils.

activity, cation exchange and pH), the ability of the soil to retain and absorb carbon and water; and reduces the fluxes of $NO_2$, $CH_4$ and $CO_2$ in the soil.

The quantities to be applied per hectare for either mosaic (e.g., farmer-managed agricultural systems with community tenure) or wide-scale (e.g., abandoned mines, plantations, state-owned forest systems) restoration approaches are yet to be determined for both the north region and Sanaga-Kadey watershed sites. However, there is a 1:3 ratio with respect to biochar use versus carbon removal and storage; i.e., 1 MT of biochar applied to agricultural plots can permanently sequesters and remove up to 3 MT/$CO_2$eq from the atmosphere [21, 22, 23, 24].

In terms of above ground capture and storage, the transition from a degraded cropping system to a tree-based system or agroforest can more than double surface carbon gains [25, 26]. If later converted to a purely tree-based system (e.g. a gallery forest) or through reforestation (e.g., in communal or community forests), this can more than triple above-ground carbon stock gains [27, 28].

Restoring the significant areas of gallery forests in the Sanaga – Kadey watersheds using mosaic (29 364 ha) and wide-scale (82 604 ha) approaches through protection, local by-laws, assisted natural regeneration and reforestation, to healthy states, can be comparable to restoring to primary or secondary mixed tropical forests. These will result in over 100,000 ha of gains in both above-ground carbon stocks being permanently stored including the associated biodiversity benefits [29, 30].

Bringing abandoned mines under restoration through a combination of ReviTech®, biochar application, reforestation and assisted natural regeneration is likely to be expensive in the short-term. However, through flexibility in managing costs (such as by using local labor and organic matter) and achieving permanence, through the development of locally useful, mixed tree systems there is good potential for below and above ground, long-term carbon capture and storage (using biochar). Through high integrity voluntary carbon capture and trading (e.g. Gold Standard or VCS) some of the financial investments can be recovered.

Finally, biochar use irrespective of the feedstock requires expensive specialized knowledge, processes and materials for chemical (elemental) analysis of the feedstock to match with the soil properties, and tailor the biochar production to the needs of soils to be brought under restoration. Dealing with assisted natural regeneration also requires selection of most appropriate tree species, determining their nutrient needs and tolerance to relevant environmental factors (local pests, drought conditions, etc.). These are areas that carbon trading can finance and create a circular, self-sustaining, win-win, process of revenue generation, environmental protection, and climate action.

### *3.4 Connectivity and biodiversity conservation*

Across rivers and streams in the Sanaga-Kadey watershed, progressive eutrophication of streams like the Bindiki river (Fig. 6) results in the reduction of the volume of water and the deterioration of its physico-chemical properties to support fish for home use by local populations. The recommendations for ecosystem restoration through development of viable woodlots, agroforests and green belts by communities (including women) will reduce siltation and help recovery of the river and freshwater life.

This setting is an ideal case for the combined application of tailor-made biochar with heavy metal absorption properties and to support both active and passive restoration (e.g., natural regeneration) for increased hydrolytic retention as well as the accumulation of minerals necessary for the growth of biodiversity on the completely eroded land. Women's groups, through robust stakeholder engagement processes, are then an effective mechanism to ensure monitoring and compliance to ensure recovery of green banks.

### *3.5 Social inclusiveness and private sector engagement*

Women are some of the victims of degrading multifunctional landscapes; such as loss of soil fertility, precarity in domestic energy resources and deterioration of water quality. Individual women have also been known to be an important part of the problem of forest landscape degradation as they fend for their families through smallholder agriculture, fuelwood harvesting, etc.

Well organized women's groups can however, play a significant role in accelerating landscape restoration. Securing the consent and participation of women's groups in envisaged ecosystem restoration programs in the Sanaga-Kadey watershed was assessed through the participator process of Free Prior and Informed Consent (FPIC). Getting their views required focused group discussions with the women, and accompanying them on field visits, to their different places of activities (Figure 8). FPIC is an immersion approach which does not only secure the consent of social groups during project implementation, but helps program proponents to develop and appreciate the motivation of social groups; understand their perception and actual states of wellbeing; the challenges they face, and their actual livelihoods strategies pertaining to how they interact with critical resources facing degradation.

*Figure 8:* Accompanying women to an artisanal gold mining site in Yassa village (Bindiba) to experience landscape degradation within the Sanaga – Kadey watershed.

Social inclusiveness in ecosystem restoration often targets the vulnerable. Therefore, one deliberate strategy to strengthen inclusiveness is to link it to engagement with the private sector, given the latter often tend s to be powerful and influential. The private sector can thus leverage social legitimacy, while the vulnerable groups can ride on the private sector power, resources and influence.

This restoration assessment thus specifically proposes an innovative human network built on a stakeholder infrastructure composed of three (03) main categories; (i) *Investors* – such as Biochar developers and carbon markets facilitators/brokers like FACET POWER[9]; (ii) *service providers* – nursery managers, trainers, seedlings distributors, researches and knowledge holders (ReviTech), and (iii) *active and passive restorers* - farmers, women's groups, planters, plantation developers.

FACET POWER seeks to produce biochar by engaging contractually in the north with SEMRI (rice company producing husks as feedstock), and with PALISCO - a timber processing company (producing wood waste as feedstock). Another potential private entity is *Terra Formations, a private* forestry regeneration and carbon markets facilitator who have expressed interest in the restoration assessment

[9] FACET power established a Memorandum of Understanding in 2022 with the WWF to take forward this initiative.

results in the Sanaga – Kadey watershed. Service providers abound and include ReviTech, small nursery managers and NGOS in Garaoua and Bertoua, or others even further afield in established institutions willing to invest resources if a business opportunity emerges, either directly or via local satellites. The third group in this innovative human network of delivery infrastructure are the active and passive restorers. Electronic (virtual) and physical market-place options which need investments and further development, have been proposed to link these three actors in real-time, dynamize their interaction and accelerate rebuilding of Cameroon's crumbling green infrastructure.

### *3.6 Developing maintenance-friendly restoration designs*

The lesson of ecosystem restoration in multifunctional, farmer- management landscapes come from the restoration practice in the forest-savanna transition zone (Box 3) of central Cameroon. This case study is the subject of a recent publication [9] which makes a strong case for sustainability and a maintenance culture to be built into restoration processes. In the time left in this decade (2020-2030) of landscape restoration, where restoration involving vulnerable groups is expected to play a major role, it is critical to have a long-term view (beyond 2030). Practitioners of ecosystem restoration must be attentive to specificities - the long-term needs of farmers (in particular, vulnerable groups), and of sensitive ecosystems under restoration. Especially, restoration must be sensitive to the physiological transformations that beneficiaries (especially rural women) would have to go through, and how these changes would influence their ability to sustain multiple, often labour intensive, maintenance activities, critical to the sustainability of restoration investments. These include phytosanitary challenges, marketing challenges; renewal of tree-based systems and general support with new knowledge and technologies, to ensure these restored landscapes remain relevant to the livelihoods of their beneficiaries. If such precautions, safeguards are ignored, restoration investments, just like traditional infrastructure, would fall into neglect and be lost, including their benefits.

The spirit of the Rio 1992 declaration on Sustainable Development (Agenda 21), should remind us of a 'common future', between project developers, promoters, investors and beneficiaries. The earlier concept, that once the restoration intervention has ended, agencies and project leaders may turn their back on green infrastructure, needs to be revisited in the same way all traditional infrastructure are only as good as their maintenance plans.

## IV. CONCLUSIONS

Green infrastructure and traditional hard infrastructure have strong parallels and understanding how the latter can be secured helps us deal better with the former. Just like traditional hard infrastructure, restoring ecosystems requires similar outcomes comprising *inter alia*; multifunctionality, climate resilience, critical resource use efficiency, carbon neutrality, connectivity, land tenure, stakeholder engagement, social inclusivity and maintenance-friendliness, as central notions.

The linkage to hard infrastructure should make ecosystem restoration less abstract to policy makers and society. It may be approached and valued at least in a similar way as society values and cares for infrastructure, not viewed only as a consumable, but as a necessary part of production and sustainability.

However, a challenge persists that, green infrastructure like biodiversity is utilitarian, is extracted and can be depleted, not a perception of traditional real-estate or hard infrastructure. However green infrastructure also regenerates and renews itself. So, one way to address this perceived deficit in green infrastructure is to use it in ways that enhance renewal, regeneration and to avoid permanent damage.

It is now urgent that restoring green infrastructure is accompanied by a look forward, a maintenance mentality, an "after-sales service". This should not be an afterthought that comes when ecosystems degrade. It should be built from the outset; in the same way maintenance is at the core of other types of infrastructure development.

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*Conflict of Interest*

The authors declare no conflict of interest.

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# Imaginary Numbers: An Absurd Starting Point or a Mathematical Necessity?

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## ABSTRACT

Since the 19th century, complex analysis, including imaginary numbers, has become a major branch of mathematics. However, can a single algebraic operation not only reveal the infallibility of imaginary numbers (complex number theory) but also destroy it? This paper aims to challenge the mathematical basis of imaginary numbers and look at their historical development from a new perspective. In this article, we demonstrate the possibility of the non-existence of imaginary numbers based on examining imaginary numbers byusing the exponential function.

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*Since the 19th century, complex analysis, including imaginary numbers, has become a major branch of mathematics. However, can a single algebraic operation not only reveal the infallibility of imaginary numbers (complex number theory) but also destroy it? This paper aims to challenge the mathematical basis of imaginary numbers and look at their historical development from a new perspective. In this article, we demonstrate the possibility of the non-existence of imaginary numbers based on examining imaginary numbers byusing the exponential function.*


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## I. INTRODUCTION

If a negative number multiplied by itself equals a positive number, then it's hard to understand the square root of a negative number. [1]

I read somewhere but cannot find the source: "When we teach complex numbers, we usually start with an absurd assumption. We define *i* to be the square root of -1. Then, we construct an elegant theory. But since we start with an absurd assumption, many people have lingering doubts. We don't have to start from an absurd point." What is this starting point of date for imaginary and complex numbers?

The imaginary numbers may be that existence is hidden in little things we don't understand.

Negative numbers were not commonly accepted among mathematicians until the late 18th century. Think about it! The Egyptians could build the pyramids, the Romans could build and maintain their huge empire. Even Newton (and Kepler and others) could work out the laws of physics and predict planetary motions, WITHOUT the concept of negative numbers. [2].

Understanding the primordial date of the beginning of an imaginary number is also an important aspect of history. Accounting back from Heron of Alexandria [3], it is 1963 years, Bhaskara Acharya [4][5] 1549 years, Girolamo Cardano [6][7] 490 years, and René Descartes [8] 398 years passed. Since that early time, the imaginary number has been studied without interruption.

In mathematics, a complex number is an element of a number system that extends the real numbers with a specific element denoted *i*, called the imaginary unit, and satisfies the equation $i^2 =- 1$; every complex number can be expressed in the form

$$a + bi \qquad [8][9] \qquad (1)$$

Where a and b are real numbers.

Because no real number satisfies the above equation, *i* was called an imaginary number by René Descartes.

We used the exponential functions for naked 1 to analyze the nature of imaginary numbers.

## II. THE IMAGINATION OF THE IMAGINARY NUMBER

The adjective imaginary was first used (as French imaginaire) by René Descartes in 1673, La Geometrie, referring to imaginary numbers in the broad sense, as non-real roots of polynomials. [8]
Euler only used the imaginary number but did not explain it. The word imaginary means that the real numbers may be slightly true in content. However, any imaginary number under the root is always different from real numbers and cannot be mixed with any other number or vanish (Identity (2) and Identity (3)).

$$\sqrt{-a} = \sqrt{a\cdot(-1)} = \sqrt{a}\cdot\sqrt{-1} \tag{2}$$

$$a + ib = a + \sqrt{-1}\cdot b \tag{3}$$

### *2.1 The Imagination of The Imaginary Number*

Expression √-1 does not exist in nature. And there is no imaginary number that has nature's innermost secrets. Human abstractions such as imaginary birds, imaginary flowers, and imaginary melodies can exist only in painting, poetry, and music.

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Euler only used the imaginary number but did not explain it. The word imaginary means that the real numbers may be true in content. However, any imaginary number under the root is always different from real numbers and cannot be mixed with any other number or vanish (Identity (2) and Identity (3)).

$$i = \sqrt{-1} \tag{4}$$

The most magic number in mathematics is the number ONE, which stands at the junction of the highest and lowest numbers on the number line.

Let's check the imaginary number using the following exponent equation (5) and the magic number 1.

$$a^{-x} = \frac{1}{a^{x}} \qquad [10] \tag{5}$$

And, if $a = 1;\ x = 1$ the intriguing problem comes in. In this case, we see the next naked identity (6):

$$1^{-1} = \frac{1}{1^{1}} = \frac{1}{1} = 1^{1} \tag{6}$$

And for 1, the next identities (7-8) are correct:

$$1^{-1} = 1^{1} \tag{7}$$

$$-1 = 1 \tag{8}$$

It is only valid when 1 is it. An imaginary number is inherently associated with the number 1.

Hence,

$$i = \sqrt{-1} = \sqrt{1} = 1 \tag{9}$$

This sounds like an incorrect solution but it is that at the end. What's true, it's true.

The very existence of imaginary numbers proves that humans create their problems! To err is human.

Great mathematician failing to come to a solution. [11].

So, I don't doubt that Identity (9) is correct. $\sqrt{-1}$ under the square root is a work of mathematicians in the dawn of mathematics. This was just such a thought experiment. From this viewpoint, simple imaginary numbers were combined with real numbers and moved into complex analysis.

In this case, Euler's Formula [12-16] is incorrect:

$$e^{x} \neq cosx + sinx$$

Either way, there is no imaginary number anywhere.

Thus, it is history that the number 1 has been able to give birth to an imaginary and complex number even for a while.

We need to know when and how to use Identities (5-9), or we get counterintuitive results for calculations of the negative number.

## III. DISCUSSION

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers $b$, $S$, and $T$ where $b > 0$, $b \neq 1$, $b^{S} = b^{T}$ if and only if $S = T$ [16][17].

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use that exponential functions are one-to-one to set the exponents equal and solve for the unknown [16][17].
My questions are

Why is it possible when b\neq1 in the case of exponential functions? Why is it not possible when $b = 1$?

1 is a real number. Why is it denied for 1?
According to one-to-one property

$$n^{S} = n^{T}, S = T$$

$$\dots\dots\dots\dots\dots\dots$$

$$2^{S} = 2^{T}, S = T$$

$$1^{S} = 1^{T}, S = T$$

2. Is there a mathematical necessity to hide the imaginary number using 1?

## IV. CONCLUSION

$1^{-1} = \frac{1}{1^{1}} = \frac{1}{1} = 1 = 1^{1}$, hence, $1^{-1} = 1^{1}$, then $\sqrt{-1} = 1$. So, we conclude that there is neither an imaginary nor a complex number.

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